Washington & Lee University.

1872,-73.

Course of Engineering, Senior Class.

NOTES

ON

Rankine's Applied Mechanics - Part I.

AND

Rankine's Civil Engineering - Parts II & III.

by

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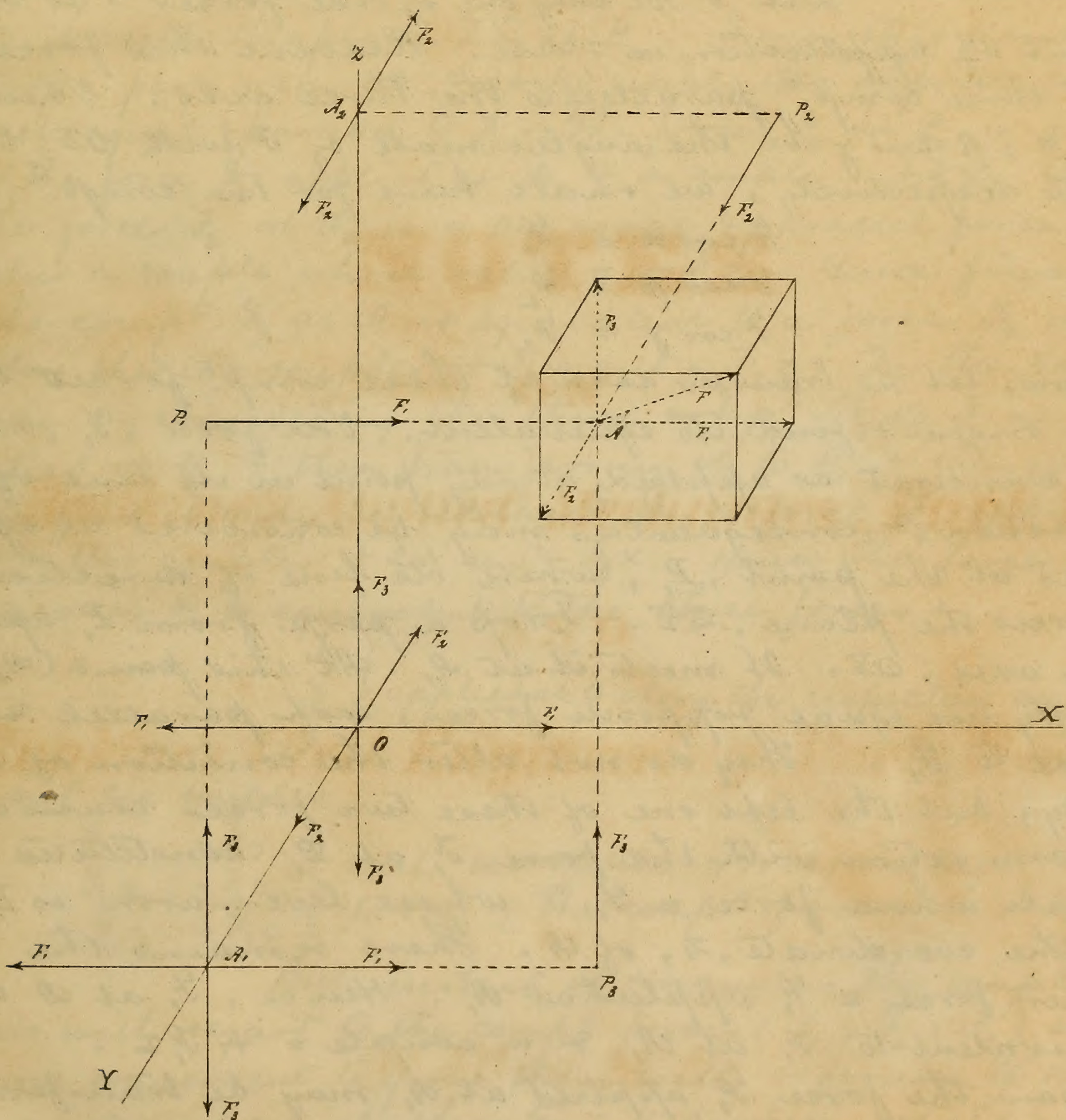
Prof. Applied Mathematics.

W. & L. U.

Baltimore, A. Hunt & Co. 1873.

Rankine's Applied Mechanics
Art. 60.

To determine the resultant of any number of forces having any directions and magnitudes & any points of application whatever in space.



This is most readily done by reffering all the forces to three coordinate axes at right angles to each other with origin at the most convenient point. Let OX, OY & OZ be these axes, & let the forces in the directions of X, Y & Z from O be considered

positive & those in the opposite directions negative. Also let the rotations which appear "righthanded" when looking from X, Y and Z towards O be considered positive (that is motion from X to Y, from Y to Z & from Z to X) & the contrary ones negative.

Let F be any one of the forces & A its point of application in space. Resolve this force into three compts parallel to the three axes. Then if α, β and γ be the angles made by F with OX, OY & OZ respectively, we shall have for the compts

$$F\cos\alpha = F_1$$
$$F\cos.\beta = F_2$$
$$F\cos\gamma = F_3$$

Now, let us transfer each of these compt forces to the origin & find its equivalent. The force, F_1, may be considered as applied at any point in its line of direction, &, consequently, may be considered as applied at the point, P_1, where its line of direction pierces the plane, ZY. Drop a perp. from P_1 upon the axis, OX. It meets it at A_1. At this point (A_1) apply two equal & opposite forces, each parallel & equal to F_1. They do not alter the condition of the body, but the left one of these two forces considered in connection with the force F_1 at P_1 constitutes a couple whose force $= F_1$ & whose lever-arm is P_1A_1 = the coordinate, z, of A. There remains the other force $= F_1$ applied at A_1. Hence, F_1 at A is equivalent to F_1 at A_1 & a couple $= + F_1 z$.

Again the force F_1 applied at A_1 may be transferred to O; for, if we apply at O two equal & opposite forces, each parallel & equal to F_1, we shall have F_1 applied at A_1 equivalent to F_1 applied at O plus a couple whose force $= F_1$ & whose lever-arm $= A_1O = y$ (one of the coordinates of A) & which is negative since it tends to produce rotation from Y towards X

This couple ∴ $= -F_1 y$. Hence to sum up. The comp.ᶜ F_1 of F parallel to OX & applied at A is equivalent to an equal & parallel force applied at the origin plus the two couples $+F_1 z$ $-F_1 y$.

Now let us take the comp.ᶜ of F in the direction of OY. This $= F_2$. Prolong its line of direction until it meets the plane ZX at P_2. From this point drop a perp. upon OZ, & at A_2 apply two opposite forces each equal & parallel to F_2. The force F_2 at P_2 is $=$ then to the force F_2 applied at A_2 & a couple $+F_2 x$. Again the force F_2 at A_2 is $=$ an equal & parallel force at O plus a couple whose value $= -F_2 z$. Hence finally the comp.ᶜ F_2 at A is equivalent to a force F_2 at O plus the two couples $+F_2 x$ and $-F_2 z$. In the same way by considering the third comp.ᶜ F_3 of F as applied at P_3 & then transferring it to A_1 & thence to O we find it may be replaced by a force F_3 applied at O & two couples $+F_3 y$ and $-F_3 x$. Hence the force F applied at A is equivalent to the three forces

$$\left.\begin{matrix} F_1 \\ F_2 \\ F_3 \end{matrix}\right\} \text{applied at } O \text{ along the respective axes}$$

and six couples

$$\left.\begin{matrix} F_1 z & -F_1 y \\ F_2 x & -F_2 z \\ F_3 y & -F_3 x \end{matrix}\right\} \begin{matrix}\text{each set of two corresponding} \\ \text{to one of the comp.}^c\text{ forces.}\end{matrix}$$

But, instead of arranging the couples in sets with regard to the comp.ᶜ forces (F_1, F_2, F_3) it is more convenient to group them with reference to the axes around which they tend to produce rotation. Grouped in this way they may be written

$F_3 y$	$-F_2 z$	tending to produce rotation around OX
$F_1 z$	$-F_3 x$	" " " " " OY
$F_2 x$	$-F_1 y$	" " " " " OZ

A similar analysis may be made for all the forces.

and by summing results we have for the resultants of any number of forces in space

$$\left.\begin{array}{l}\Sigma F\cos\alpha = \Sigma F_1 = R_1 = \text{resultant of components along } OX \\ \Sigma F\cos\beta = \Sigma F_2 = R_2 = \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad OY \\ \Sigma F\cos\gamma = \Sigma F_3 = R_3 = \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad OZ\end{array}\right\}(1)$$

and

$$\left.\begin{array}{l}\Sigma(F\cos\gamma . y - F\cos\beta . z) = \Sigma(F_3 y - F_2 z) = M_1 = \text{sum of couples about } OX \\ \Sigma(F\cos\alpha . z - F\cos\gamma . x) = \Sigma(F_1 z - F_3 x) = M_2 = \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad OY \\ \Sigma(F\cos\beta . x - F\cos\alpha . y) = \Sigma(F_2 x - F_1 y) = M_3 = \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad OZ\end{array}\right\}(2)$$

The above resultant forces, R_1, R_2, R_3, acting at O may be combined into one force (the diag. of the parallelopipedon constructed on them), & the value of this final resultant will be

$$R = \sqrt{R_1^2 + R_2^2 + R_3^2} \text{ - - - - - - - - } (4)$$

and if α_r, β_r, γ_r, be the angles it makes with the axes,

$$\cos\alpha_r = \frac{R_1}{R}, \quad \cos\beta_r = \frac{R_2}{R}, \quad \cos\gamma_r = \frac{R_3}{R}$$

These eq.s make known the magnitude & direction of the resultant force, its point of application being O.

Similarly, the three couples M_1, M_2, M_3 may be combined into one whose value is

$$M = \sqrt{M_1^2 + M_2^2 + M_3^2} \text{ - - - - - - - - } (5)$$

and if λ, μ, ν are the angles made by the axis of the resultant couple, M, with the axes of the component couples (which are the coordinate axes themselves) we shall have

$$\cos\lambda = \frac{M_1}{M}, \quad \cos\mu = \frac{M_2}{M}, \quad \cos\nu = \frac{M_3}{M}$$

Thus the value of the resultant couple and the direction of its axis are determined.

We thus see that any number of forces acting in space may be replaced by a single force $= R$ acting at the origin, whose magnitude & direction are determined, plus a couple M also of determinate magnitude

whose plane of action is known, since it is perp. to its axis.

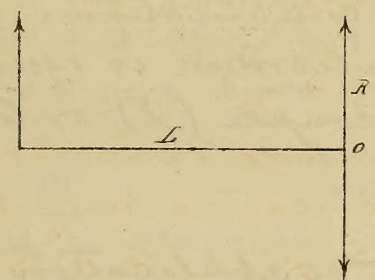
The conditions of equilibrium of such a system of forces are

$$R = 0\ , \quad M = 0$$

When the system is not balanced its resultant must fall under one of the five following cases –

1° When $M = 0$, the resultant is the single force R, acting at O.

2° When the axis of M is at right angles to the direction of R, then M acts in a plane identical with or parallel to that in which R acts. If the plane of M is parallel to that of R, we may consider the couple as transferred to the plane of R, and then we have a couple & a force in one plane to deal with. If the forces of this couple are not each $= R$, change it into an equivalent couple with forces $= R$, & move the couple so that one of its forces shall be applied at O & opposite to R. Then it is plain from the figure that the couple & force in one plane are equivalent to a single force $= R$ applied at a distance from $O = L = \frac{M}{R}$, to the left if the couple is righthanded & to the right if the couple is left-handed.



The condition rendering the axis of M perp. to R is

$$\cos\alpha_r \cos\lambda + \cos\beta_r \cos\mu + \cos\gamma_r \cos\nu = 0$$

3° When $R = 0$, the only resultant is the couple M & we have rotation without translation.

4° When the axis of M is parallel to R, then the couple M acts in a plane perp. to R & there can be no farther reduction. The body under such a system of forces will rotate in a plane perp. to the line of direction in which it moves forward (like a rifle ball). The condition rendering the axis of M parallel to R is

$\lambda = \alpha_r$ or $= -\alpha_r$; $\mu = \pm \beta_r$; $\nu = \pm \gamma_r$.

5° When the axis of M is oblique to R making with it an angle given by the eq.

$$\cos\theta = \cos\lambda\cos\alpha_r + \cos\mu\cos\beta_r + \cos\nu\cos\gamma_r \; .$$

The couple M may be resolved into two components; viz- $M\sin\theta$ around an axis perp. to R & in a plane containing the direction of R & of the axis of M, & $M\cos.\theta$ around an axis parallel to R: The force R & the couple, $M\sin\theta$, are equivalent, as in Case (2), to a single force equal & parallel to R whose line of action is in a plane perp. to that containing R & the axis of M, & whose perp. distance from O is

$$L = \frac{M\sin\theta}{R}$$

The couple, $M\cos\theta$, which acts in a plane perp. to R is incapable of further combination. Hence every system of forces not in equilibrium is equivalent to a single force (1) & (2)) or to a couple (3) or to a force & a couple ((4) & (5)).

Cor. I Suppose the <u>points of application</u> of the forces to be all in one plane - that of XY, for example. Then eq.^{s} (1) & (4) remain unchanged - that is -

$$R_1 = \Sigma.F\cos\alpha = \Sigma.F_1$$
$$R_2 = \Sigma.F\cos\beta = \Sigma F_2 \qquad R = \sqrt{R_1^2 + R_2^2 + R_3^2}$$
$$R_3 = \Sigma.F\cos\gamma = \Sigma F_3$$

But in eq.^{s} (2) the coordinates, z, $= 0$ or

$$M_1 = \Sigma(F\cos\gamma.y) = \Sigma.F_3 y$$
$$M_2 = -\Sigma(F\cos\gamma.x) = -\Sigma F_3 x$$
$$M_3 = \Sigma(F\cos\beta.x - F\cos\alpha.y) = \Sigma(F_2 x - F_1 y)$$

and

$$M = \sqrt{M_1^2 + M_2^2 + M_3^2}$$

Cor. II Suppose the forces all lie <u>in one plane</u>.

that of XY, for example – then there are no components along OZ & no coordinates, z, Eq.s (1) then become

$$\begin{aligned} R_1 &= \Sigma . F\cos\alpha = \Sigma F_1 \\ R_2 &= \Sigma . F\cos\beta = \Sigma F_2 \\ R_3 &= 0 \\ R &= \sqrt{R_1^2 + R_2^2} \end{aligned}$$

$$\cos\alpha_r = \frac{R_1}{R} \; ; \; \cos\beta_r = \frac{R_2}{R} = \sin\alpha_r$$

& eq.s (2) become

$$M_1 = 0 \; ; \; M_2 = 0 \; ; \; M_3 = \Sigma(F\cos\beta . x - F\cos\alpha . y)$$
$$= \Sigma(F_2 x - F_1 y) = M$$

The force, R, & couple, M, being in the same plane, may be combined, as in Case (2), by giving the couple a lever-arm $= \frac{M}{R} = L$. In this case then the forces are finally equivalent to a single force applied at a distance from $O = L$ & to the left or right according as the couple is right or left-handed. If $R = 0$ & M is not $= 0$, we have rotation simply. If $M = 0$ & R not $= 0$, we have a single force acting thro. the origin

Cor. III. Suppose the forces to be applied at any points in space but to be parallel to each other. Then the angles they make with the coordinate axes are constant & eq.s (1) & (2) become

$$\left.\begin{aligned} R_1 &= \cos\alpha . \Sigma F = \Sigma F_1 \\ R_2 &= \cos\beta . \Sigma F = \Sigma F_2 \\ R_3 &= \cos\gamma . \Sigma F = \Sigma F_3 \end{aligned}\right\} \text{------} (14)$$

$$\left.\begin{aligned} M_1 &= \cos\gamma . \Sigma(Fy) - \cos\beta . \Sigma(Fz) = \Sigma(F_3 y - F_2 z) \\ M_2 &= \cos\alpha . \Sigma(Fz) - \cos\gamma . \Sigma(Fx) = \Sigma(F_1 z - F_3 x) \\ M_3 &= \cos\beta . \Sigma(Fx) - \cos\alpha . \Sigma(Fy) = \Sigma(F_2 x - F_1 y) \end{aligned}\right\} \text{---} (15)$$

If we assume the coordinate axes so that OZ shall be parallel to the direction of the forces, these eq.s become (since $\cos\alpha = 0$; $\cos\beta = 0$; $\cos\gamma = 1$.)

$$R_1 = 0 \quad ; \quad R_2 = 0 \quad ; \quad R_3 = \Sigma F_3 = \Sigma F = R$$ and

$$\mathcal{M}_1 = \Sigma.\, F_3 y = \Sigma.\, Fy$$
$$\mathcal{M}_2 = -\Sigma.\, F_3 x = -\Sigma.\, Fx$$
$$\mathcal{M}_3 = 0$$
$$\therefore \quad \mathcal{M} = \sqrt{\mathcal{M}_1^2 + \mathcal{M}_2^2}$$

and if θ = angle the axis of $\mathcal{M}$ makes with OX, $\cos\theta = \frac{\mathcal{M}_1}{\mathcal{M}}$.

Cor. IV. If we assume the points of application of the parallel forces to be all in the plane, XY, then eq.s (14) & (15) become

$$R_1 = \cos\alpha.\, \Sigma F = \Sigma F_1$$
$$R_2 = \cos\beta \cdot \Sigma F = \Sigma F_2$$
$$R_3 = \cos\gamma \cdot \Sigma F = \Sigma F_3$$
$$\mathcal{M}_1 = \cos\gamma.\, \Sigma(Fy) = \Sigma(F_3 y)$$
$$\mathcal{M}_2 = -\cos\gamma.\, \Sigma(Fx) = -\Sigma(F_3 x)$$
$$\mathcal{M}_3 = \cos\beta.\, \Sigma(Fx) - \cos\alpha.\, \Sigma(Fy) = \Sigma(F_2 x - F_1 y)$$

(This is the case of Art. 91. page. 74 when $R = 0$)

If the plane is perp. to the direction of the forces

$$R_1 = 0 \quad ; \quad R_2 = 0$$
$$R_3 = \Sigma F_3 = \Sigma F = R. \quad \text{Also}$$
$$\mathcal{M}_1 = \Sigma(Fy) \quad ; \quad \mathcal{M}_2 = -\Sigma(Fx)$$
$$\mathcal{M}_3 = 0$$

(This is the case of Art 92 when $R = 0$)

Cor. V Suppose the parallel forces all to lie in one plane; that of XY, for example. Then

$$R_1 = \cos\alpha.\, \Sigma F = \Sigma F_1$$
$$R_2 = \cos\beta.\, \Sigma F = \sin\alpha.\, \Sigma F = \Sigma F_2$$
$$R_3 = 0$$
$$R = \sqrt{R_1^2 + R_2^2}$$

Also $\mathcal{M}_1 = 0 \quad ; \quad \mathcal{M}_2 = 0$

$$\mathcal{M}_3 = \cos\beta.\, \Sigma(Fx) - \cos\alpha.\, \Sigma(Fy) = \Sigma(F_2 x - F_1 y) = \mathcal{M}$$

If the axis Y be taken parallel to the direction

of the forces, R_1 will be zero, & we will have

$$R_2 = \Sigma F = R$$
$$M_3 = M = \Sigma(Fx).$$

In corollaries III, IV & V, since the forces are parallel, the planes of action of R & M must be identical or parallel, & the force & couple can therefore be combined; & hence the ultimate resultant of all sets of parallel forces is a force equal to their algebraic sum (= R) applied at a distance from the origin $= \bar{x} = \frac{M}{R}$. (See Case 2.)

If in any case R = 0 while M has a finite value, the resultant is merely the couple M.

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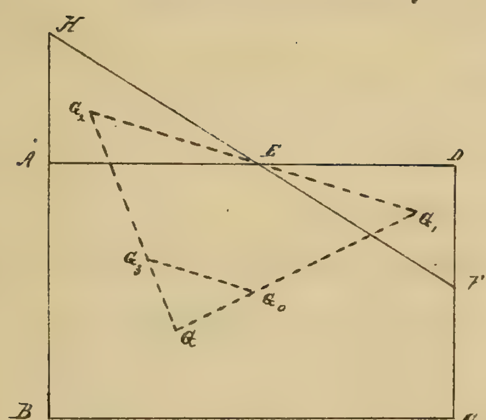
App. Mech. Art. 77.



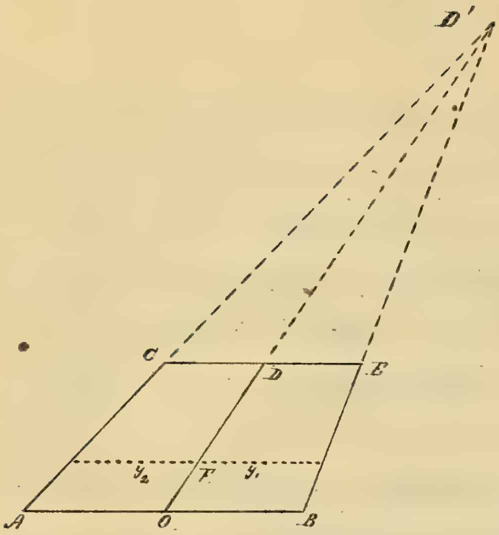
By parallel forces, the centre of gr. of BCFEA or W at G must be in the prolongation of the line G_1G_0, since a force $= W_0$ at G_0 may be considered as balancing that at G & that at G_1. For same reason, G_3 (centre of grav. of BCFEH) must be in the line GG_2. Also

$$W_0 : W_1 : W :: GG_1 : GG_0 : G_0G_1$$
$$W_0 : W_1 : W :: GG_2 : GG_3 : G_2G_3$$
$$\therefore\ GG_1 : GG_0 :: GG_2 : GG_3 :: G_1G_2 : G_0G_3$$
$$\therefore\ W_0 : W_1 :: G_1G_2 : G_0G_3$$
$$\therefore\ G_0G_3 = G_1G_2 \cdot \frac{W_1}{W_0}$$

(G_0G_3 is parallel to G_1G_2)

A. M. Art. 83.

Ex. III.



Prolong the sides until they meet at D'. Then the origin being at O with OD & OB for axes, we have by formula (4), Art. 78, the co-ordinates of the C. of G. in direction OD,

$$x_0 = \frac{\int x(y_2+y_1)\,dx \cdot \sin DOB}{\int (y_2+y_1)\,dx \cdot \sin DOB} \quad \text{or}$$

$$x = \frac{\int x(y_2+y_1)\,dx}{\int (y_2+y_1)\,dx}.$$

Since $AB = B$ & $CE = b$, we have by similar triangles

$$B : b :: OD' : DD' \quad \text{or} \quad B : b :: OD' : OD' - OD$$

$$\therefore\ OD' = OD\,\frac{B}{B-b}$$

Again

$$B : y_2 + y_1 :: OD' : FD'$$

$$B : y_2 + y_1 :: OD' : OD' - x$$

$$\therefore\ B(OD' - x) = OD'(y_2 + y_1)$$

Substituting the value of OD' above, we have

$$y_2 + y_1 = \frac{B\left(OD\frac{B}{B-b} - x\right)}{OD\left(\frac{B}{B-b}\right)} = \frac{OD \cdot B - x(B-b)}{OD} = B - x\,\frac{B-b}{OD}$$

Substitute this in the expression for x_0 & integrate. Then

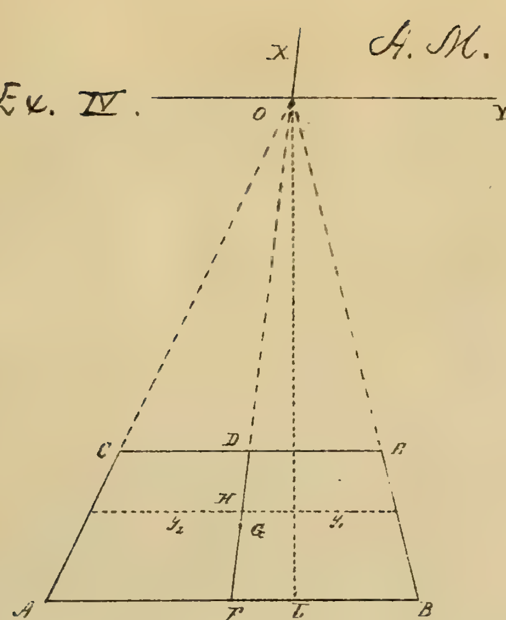
$$x_0 = \frac{\int_0^{OD} x\left(B - x\frac{B-b}{OD}\right)dx}{\int_0^{OD}\left(B - x\frac{B-b}{OD}\right)dx} = \frac{\frac{Bx^2}{2} - \frac{1}{3}\cdot\frac{B-b}{OD}\cdot x^3}{Bx - \frac{1}{2}\cdot\frac{B-b}{OD}\cdot x^2} + C$$

$$= \frac{x}{2} - \frac{\frac{1}{12}\cdot\frac{B-b}{OD}\cdot x^3}{Bx - \frac{1}{2}\cdot\frac{B-b}{OD}\cdot x^2} + C = \frac{OD}{2} - \frac{1}{6}\cdot\frac{B-b}{B+b}\cdot OD$$

The C. of G. being on OD, the coordinate, y_0, $= 0$

A. M. Art. 83.

Ex. IV.



Let O be the origin & OF & OY the axes making an angle = OFB. Then, use the eq.

$$x_0 = \frac{\int x(y_2+y_1)\,dx}{\int (y_2+y_1)\,dx}$$

Let OH = x. Then

$y_2+y_1 : x :: AB : x_1$

$\therefore\ y_2+y_1 = AB\cdot\frac{x}{x_1}$

Hence $x_0 = \frac{\int x^2 dx}{\int x\,dx} = \frac{2}{3}\cdot\frac{x_1^3-x_2^3}{x_1^2-x_2^2}$

bet. the limits OD & OF.

Again, from eq.(2). Art. 78, A.M.

$W = wz\int(y_2+y_1)\,dx\,.\sin OFB$ But $z=1$

$\therefore\ W = w\int\frac{AB}{x_1}.\,x\,dx.\sin OFB$

$= w.\sin OFB.\,\frac{AB}{x_1}.\,\frac{x_1^2-x_2^2}{2}$ - - - - - - - (2)

$AL = OL.\cot OAF$; $LB = OL\cot OBF$.

$\therefore\ AB = OL(\cot OAF + \cot OBF)$

But $OL = OF\sin OFB$; & since x_1 also $= OF$, we have from (2)

$$W = w\sin^2 OFB.\,\frac{x_1^2-x_2^2}{2}(\cot OAF + \cot OBF)$$

Similarly we find value of x_0W.

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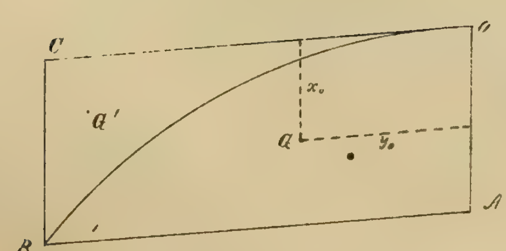
A. M. Art. 83.

Ex. V.



Here O is the origin & AO & OC the axes. Then from eq. 4, Art. 78. A. M. we have

$$x_0 = \frac{\int xy\,dx}{\int y\,dx}\ ;\quad y_0 = \frac{\int y^2 dx}{2\int y\,dx}$$

since $y^2 = 2px$

$$x_0 = \frac{\int \sqrt{2p}\cdot x^{3/2}\,dx}{\int \sqrt{2p}\cdot x^{1/2}\,dx} = \frac{\frac{2}{5}x^{5/2}}{\frac{2}{3}x^{3/2}} = \frac{3}{5}x_1$$

and

$$y_0 = \frac{\int 2px\,dx}{2\int \sqrt{2p}\cdot x^{1/2}dx} = \frac{\sqrt{2p}\cdot\frac{x^2}{2}}{2\cdot\frac{2}{3}x^{3/2}} = \frac{3}{8}\sqrt{2px} = \frac{3}{8}y_1$$

— " —

A. M. Art. 83.

Ex. VI.

Here the spandril may be considered as gotten by subtracting the half-segment from the parallelogram OCBA; & if x_3 and y_3 be the coordinates of G', its C. of G., we will have

$$(W_1 - W_2)x_3 = W_1 x_1' - W_2 x_2' = W_1\frac{x_1}{2} - \frac{2}{3}W_1\cdot\frac{3}{5}x_1$$

$$\therefore\quad x_3 = \frac{W_1\cdot\frac{x_1}{2} - \frac{2}{3}\cdot W_1\cdot\frac{3}{5}x_1}{W_1 - \frac{2}{3}W_1} = \frac{\frac{1}{10}W_1 x_1}{\frac{1}{3}W_1} = \frac{3}{10}x_1 .$$

y_3 is obtained similarly.

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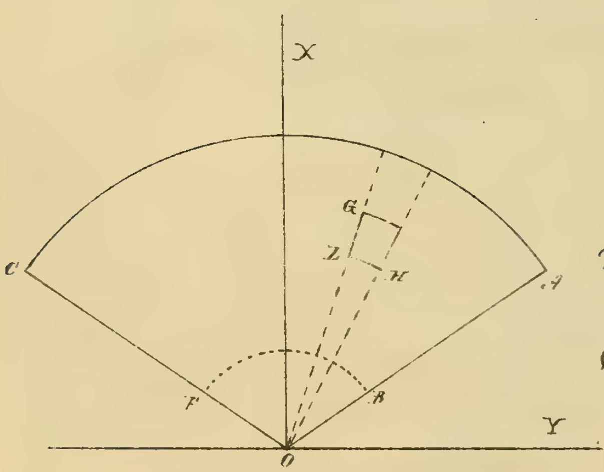
A. M. Art. 83.

Ex. VII.



The sector may be considered as made up of little wedges like GH, the area of which = HZ. GZ = $r\,dr\,d\theta$. The leverage of this about OY = x. = $r\cos\theta$ (counting θ from OX). Hence

$$x_0 = \frac{\int_{r=r_1}^{r=0}\int_{\theta=\theta_1}^{\theta=-\theta_1} r^2\cos\theta\,dr\,d\theta}{\int_{r=r_1}^{r=0}\int_{\theta=\theta_1}^{\theta=-\theta_1} r\,dr\,d\theta}$$

$$= \frac{\int\frac{r^3}{3}\cos\theta\,d\theta}{\int\frac{r^2}{2}\cdot d\theta} = \frac{2}{3}r_1\frac{\sin\theta_1}{\theta_1}$$

$$y_0 = \frac{\int_{r=r_1}^{r=0}\int_{\theta=\theta_1}^{\theta=-\theta_1} r^2 \sin\theta \, dr \, d\theta}{\int_{r=r_1}^{r=0}\int_{\theta=\theta_1}^{\theta=-\theta_1} r \, dr . d\theta} = \frac{2}{3} r_1 \frac{\int_{\theta=\theta_1}^{\theta=-\theta_1} \sin\theta . d\theta}{\int_{\theta=\theta_1}^{\theta=-\theta_1} d\theta}$$

$$= \frac{2}{3} r_1 . \frac{(\cos\theta_1 - \cos\theta_1)}{\theta_1 - \theta_1} = 0$$

If we take the ring, FCAB, instead of the sector, we have for the limits of r, $r = r_1' = OB$, & $r = r_1 = OA$, & then

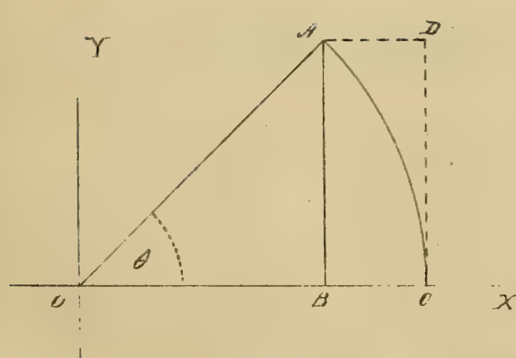
$$x_0 = \frac{\int_{r=r_1}^{r=r_1'}\int_{\theta=\theta_1}^{\theta=-\theta_1} r^2 \cos\theta . dr . d\theta}{\int_{r=r_1}^{r=r_1'}\int_{\theta=\theta_1}^{\theta=-\theta_1} r \, dr \, d\theta} = \frac{2}{3} \frac{r_1^3 - r_1'^3}{r_1^2 - r_1'^2} . \frac{\sin\theta_1}{\theta_1}$$

$$y_0 = 0$$

This is Ex. X. in the text.

A. M. Art. 83.

Ex. VIII



Ex. VIII can be solved by subtraction—thus—the C. of G. of the sector, OCA, gives

$$x_0 = \frac{\iint r^2 \cos\theta \, dr \, d\theta}{\iint r \, dr \, d\theta} = \frac{2}{3} r . \frac{\sin\theta}{\theta}$$

$$y_0 = \frac{\iint r^2 \sin\theta . dr \, d\theta}{\iint r \, dr \, d\theta} = \frac{2}{3} r \frac{(1 - \cos\theta)}{\theta}$$

$$= \frac{2}{3} r . \frac{2 \sin^2 \frac{\theta}{2}}{\theta}$$

& that of triangle, OBA,

$$x_1 = \frac{2}{3} r \cos\theta \ ; \quad y_1 = \frac{1}{3} r \sin\theta$$

$$W_0 = OAC \ ; \quad W_1 = OBA$$

Then the coordinates of the C. of G. of the half segment ABC (x_0', y_0') are

$$x_0' = \frac{W_0 x_0 - W_1 x_1}{W_0 - W_1} = \frac{2}{3} r_1 \frac{\sin^3\theta}{\theta - \sin\theta\cos\theta} \;;\; y_0' = \frac{W_0 y_0 - W_1 y_1}{W_0 - W_1}$$

$$\therefore\; y_0' = \frac{1}{3} r \cdot \frac{4\sin^2\frac{\theta}{2} - \sin^2\theta\cos\theta}{\theta - \sin\theta\cos\theta}$$

A. M. Art. 83.

Ex. IX.

The figure in the last example will do by completing the rect. ADCB. Then the C. of G. may be found by subtraction. Thus, let W_1 = wt. of rect. & W_2 = that of half-segment ABC. The coordinates of the C. of G. of the rect. are

$$x_0' = OB + \frac{BC}{2} = a + \frac{r-a}{2} = \frac{a+r}{2} = \frac{r(1+\cos\theta)}{2}$$

$$y_0' = \frac{AB}{2} = \frac{r\cdot\sin\theta}{2}$$

Again, the area of the rect. $= \overline{AB}\cdot\overline{BC} = b(r-a) = r\sin\theta\,(r - r\cos\theta)$.

Area of segment (Ex. VIII) $= \frac{1}{2}r^2(\theta - \cos\theta\sin\theta)$

Hence $$(W_1 - W_2)x_0 = W_1\left(\frac{a+r}{2}\right) - W_2\left(\frac{2}{3}r\cdot\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\right)$$

$$\therefore\; x_0 = \frac{r\cdot\frac{(1+\cos\theta)}{2}\cdot r^2\sin\theta\,(1-\cos\theta) - \frac{2}{3}r\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\cdot\frac{1}{2}r^2(\theta - \sin\theta\cos\theta)}{r^2\sin\theta\,(1-\cos\theta) - \frac{1}{2}r^2(\theta - \sin\theta\cos\theta)}$$

$$= \frac{\frac{r}{2}(1-\cos^2\theta)\sin\theta - \frac{1}{3}r\cdot\sin^3\theta}{\sin\theta - \frac{1}{2}\sin\theta\cos\theta - \frac{1}{2}\theta} = \frac{1}{3}r\frac{\sin^3\theta}{2\sin\theta - \sin\theta\cos\theta - \theta}$$

So $$(W_1 - W_2)y_0 = W_1\frac{b}{2} - W_2\left(r\frac{4\sin^2\frac{\theta}{2} - \sin^2\theta\cos\theta}{3(\theta - \cos\theta\sin\theta)}\right)$$

$$\therefore\; y_0 = \frac{\frac{r^3\sin^2\theta\,(1-\cos\theta)}{2} - r\left(\frac{4\sin^2\frac{\theta}{2} - \sin^2\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right)\frac{r^2}{2}(\theta - \sin\theta\cos\theta)}{r^2\sin\theta\,(1-\cos\theta) - \frac{1}{2}r^2(\theta - \sin\theta\cos\theta)}$$

$$= \frac{r(\sin^2\theta - \sin^2\theta\cos\theta - \frac{4}{3}\sin^2\frac{\theta}{2} + \frac{1}{3}\sin^2\theta\cos\theta)}{2\sin\theta - \sin\theta\cos\theta - \theta}$$

$$= \frac{r}{3}\,\frac{3\sin^2\theta - 2\sin^2\theta\cos\theta - 4\sin^2\frac{\theta}{2}}{2\sin\theta - \sin\theta\cos\theta - \theta}$$

A. M. Art. 83.

Ex. XV. By formula for wedges, A. M. Art. 83, we have

$$x_0 = \frac{\int x^2 y\, dx}{\int xy\, dx} = \frac{\int x^2(r^2-x^2)^{1/2}dx}{\int x(r^2-x^2)^{1/2}dx}.$$

Applying formula (A), we get

$$\int x^2(r^2-x^2)^{1/2}dx = \frac{x(r^2-x^2)^{3/2} - r^2\int(r^2-x^2)^{1/2}dx}{4}$$

Applying formula (B) to this last integral, we have

$$\int(r^2-x^2)^{1/2}dx = \frac{x(r^2-x^2)^{1/2} + r^2\int(r^2-x^2)^{-\frac{1}{2}}dx}{2}.$$

$$\int(r^2-x^2)^{-\frac{1}{2}}dx = \int\frac{dx}{r\left(1-\frac{x^2}{r^2}\right)^{1/2}} = \sin^{-1}\frac{x}{r} + C$$

Adding these partial integrals, we get for the total integral

$$\int_r^0 x^2(r^2-x^2)^{1/2}dx = \frac{x(r^2-x^2)^{3/2} - r^2\left(\frac{x(r^2-x^2)^{1/2} + r^2\sin^{-1}\frac{x}{r}}{2}\right)}{4} = -\frac{\pi r^4}{16}$$

Integrating the denominator of the value of x_0, we have

$$\int_r^0 x(r^2-x^2)^{1/2}dx = -\frac{1}{3}(r^2-x^2)^{3/2} = -\frac{1}{3}r^3.$$

$$\therefore x_0 = \frac{-\frac{\pi r^4}{16}}{-\frac{r^3}{3}} = \frac{3}{16}\pi r$$

Ex. VI is solved in the same way by using diff. limits.

— " —

A. M. Art. 83

Ex. XVIII.

This example may be solved by subtraction.

Let W_1 = wt. of large pyramid.

" W_2 = " " pyr. to be subtracted.

" W_3 = " " frustum.

$$W_3 x_3 = W_1 x_1 - W_2 x_2 \quad \therefore \quad x_3 = \frac{x_1 W_1 - x_2 W_2}{W_1 - W_2}$$

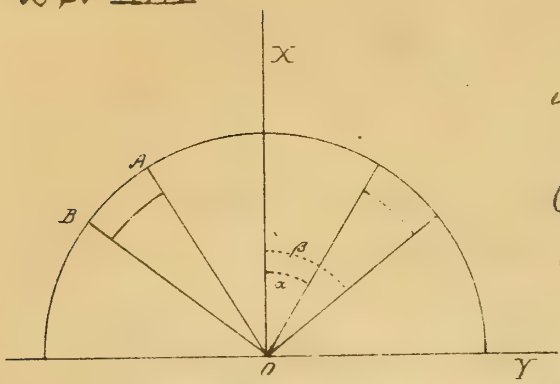
and since

$$W_1 : W_2 :: h^3 : h'^3 \quad \therefore \quad W_2 = \frac{W_1 h'^3}{h^3}$$

$$\therefore x_3 = \frac{\frac{3}{4}hW_1 - \frac{3}{4}h'\frac{W_1 h'^3}{h^3}}{W_1 - W_1\frac{h'^3}{h^3}} = \frac{3}{4}\,\frac{h^4 - h'^4}{h^3 - h'^3}$$

A. M. Art. 83.

Ex. XIX



The general formula for the integration of such a solid is

$$V = \iiint r^2 \sin\theta \cdot dv\, dr\, d\theta$$

Count θ from X towards B. The leverage of the infinitesimal solid about OY = $r\cos\theta$. Hence

$$x_0 = \frac{\iiint r^3 \sin\theta \cos\theta .\, dv .\, dr .\, d\theta}{\iiint r^2 \sin\theta\, dv\, dr\, d\theta}.$$

The limits of integration are

$v = 2\pi$ & $v = 0$; $r = r$ & $r = r'$; $\theta = \alpha$ & $\theta = \beta$.

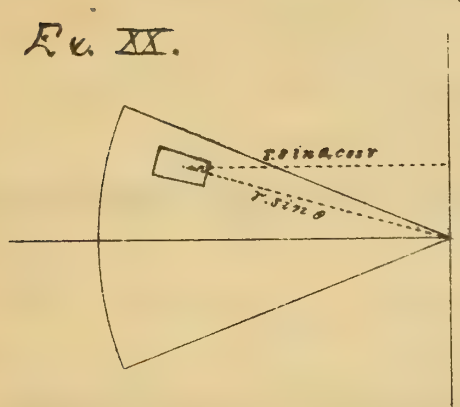
Between these limits

$$x_0 = \frac{2\pi \cdot \frac{r^4 - r'^4}{4} \cdot \frac{\cos^2\beta - \cos^2\alpha}{2}}{2\pi \cdot \frac{r^3 - r'^3}{3} \cdot (\cos\beta - \cos\alpha)} = \frac{3}{4}\, \frac{r^4 - r'^4}{r^3 - r'^3} \cdot \frac{\cos\beta + \cos\alpha}{2}$$

The denominator of the value of x_0 = W.

A. M. Art. 83.

Ex. XX.



Use the same formula but diff. limits. These limits now are

$v = +v'$ & $v = -v'$; (This angle = θ of Rank.)

$r = +r$ & $r = +r'$; $\theta = 0°$ & $\theta = 90°$

Lever-arm about plane perp. to OX = $r\cos\theta$. That about plane perp to OY = $r\sin\theta\cos v$. Hence

$$x_0 = \frac{\iiint r\cos\theta .\, r^2 \sin\theta .\, dr\, d\theta\, dv}{\iiint r^2 \sin\theta\, d\theta\, dr\, dv} = \frac{2v' . \frac{r^4 - r'^4}{4} . \cos^2\theta}{2v' . \frac{r^3 - r'^3}{3} . \cos\theta} = \frac{3}{4} . \frac{r^4 - r'^4}{r^3 - r'^3}$$

$$y_0 = \frac{\iiint r \sin\theta \cos v .\, r^2 \sin\theta\, d\theta\, dr\, dv}{\iiint r^2 \sin\theta\, d\theta\, dr\, dv} = \frac{2 \sin v' \frac{r^4 - r'^4}{4} . \left(\frac{\cos\theta \sin\theta}{2} + \frac{\theta}{2}\right)}{2v' . \frac{r^3 - r'^3}{3} . \cos\theta}$$

$$= \frac{\frac{1}{4} \sin v' \, \frac{r^4 - r'^4}{4} . \pi}{v' \, \frac{r^3 - r'^3}{3}} = \frac{3}{16} . \frac{\pi (r^4 - r'^4)}{r^3 - r'^3} . \frac{\sin v'}{v'}$$

A. M. Art. 94.

—"—

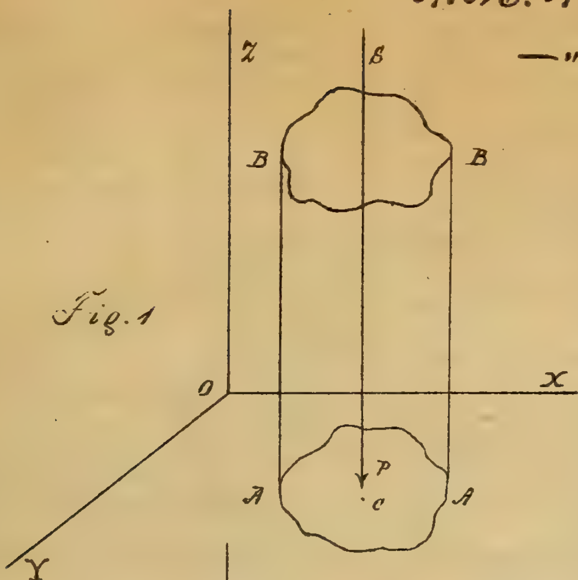


Fig. 1

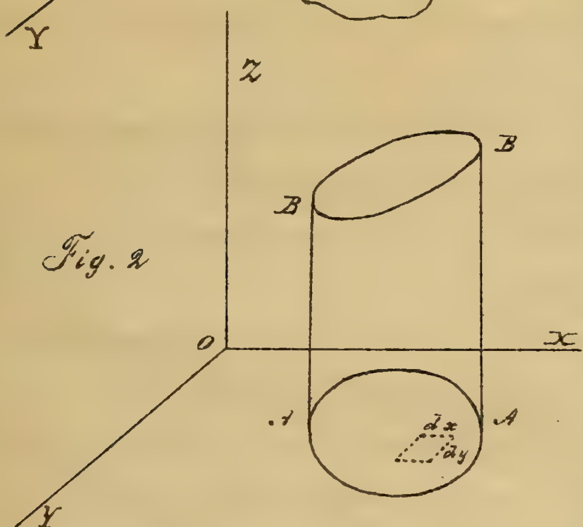


Fig. 2

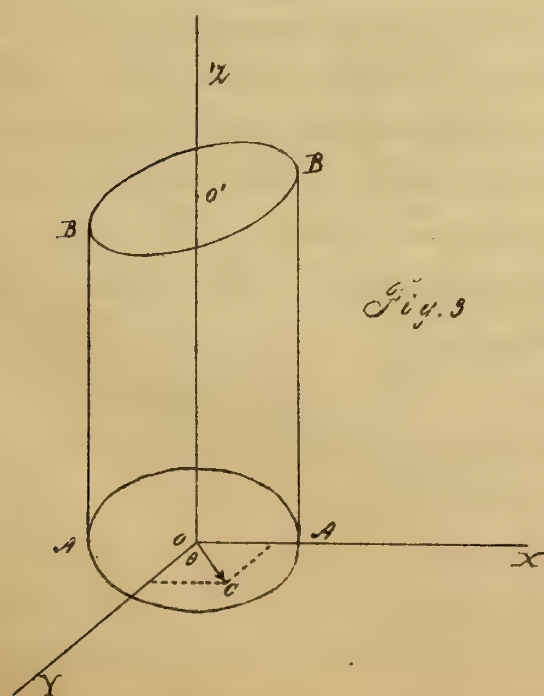


Fig. 3

Any varying external force distributed over the surface AA may be represented as in Fig. 1. by a cylinder raised upon the surface, & bounded at the upper end BB by a surface whose ordinate z at each point shall be proportional to the intensity of the pressure at the foot of the ordinate. Now, if the resultant of this external distributed force (= P) pass thro. C, that point is the centre of pressure, & the line CS evidently passes thro. the C. of G. of the cylinder.

The internal stress developed in the surface AA by the pressure P must = P & must have a resultant passing thro. C. It too may practically be represented by the cylinder AABB.

The pressure on AA may be uniform (in which case the cylinder becomes a prism) or it may vary in any way, the character of the varaition being indicated by the shape of the surface BB. The only kind of varying pressure or stress we need discuss is uniformly varying in which the surface BB is a plane inclined to the base AA.

Pressure or stress, such as we are considering, is but a sys-

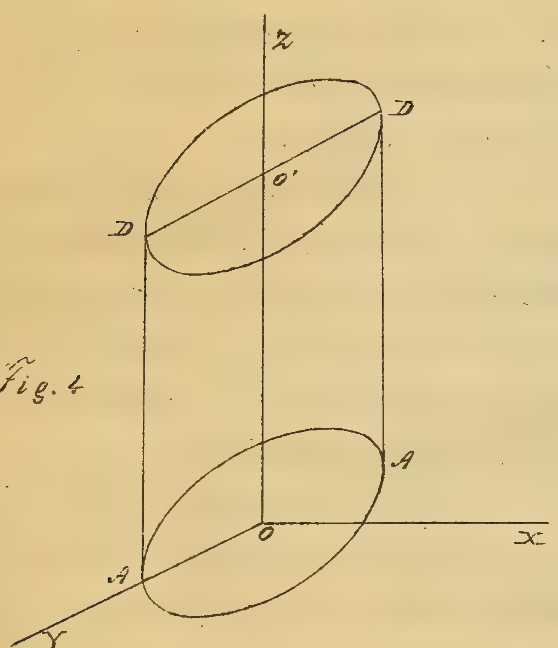
tem of parallel forces. If we let p = unit of pressure or stress & $dy.dx$ = differential of the surface AA, then total amt. of stress = $P = \iint p\,dx.dy$.

So too the centre of pressure, which is merely the centre of parallel forces, has for its coordinates

$$x_0 = \frac{\iint px.dx\,dy}{\iint p\,dx\,dy} \quad \text{---}(1). \qquad y_0 = \frac{\iint py\,dx.dy}{\iint p\,dx\,dy} \quad \text{---}(2)$$

In uniformly varying stress these formulae may be simplified by taking the origin at O, the C. of G. of the surface AA, & by taking the axis of y in the same direction as that line in the surface BB which is parallel to the plane AA.

Fig. 4

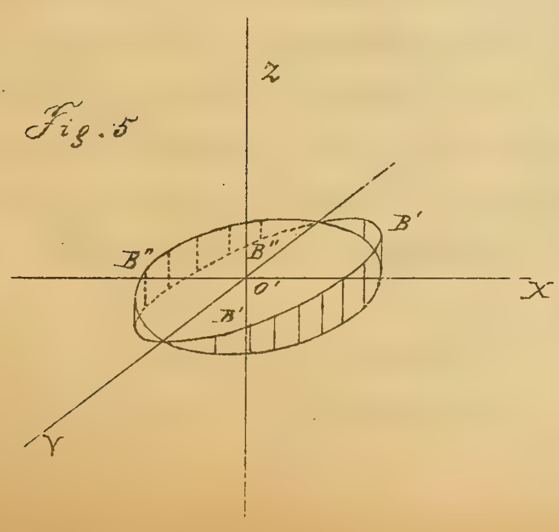


When this has been done conceive the cylinder representing the stress to be composed of two parts –

1° A cylinder with upper base parallel to AA & a height = OO' = mean height of cylinder AB. This cylinder is seen in fig. 4. & is = AB (Fig. 3) in volume.

2° Two equal wedges O'B'B' & O'B"B", the one above & the other below the plane passing thro. O' parallel to AA – the one above the plane to be added to the cylinder just described & the other to be subtracted.

Fig. 5



These two solids together are equal to the original cylinder AB (Fig. 3). The cylinder AD (Fig. 4.) represents a uniform force whose amt., as above stated, = P & whose intensity = $\frac{P}{S} = p_0$. The wedges (Fig. 5) represent a uniformly varying

force whose neutral axis = OY, & whose amt. = zero, & whose intensity at each point = $p' = ax$. Hence the total intensity of pressure at any point of AA is

$$p = p_0 + p' = p_0 + ax.$$

where x changes sign when to the left of OY.

Substituting this value of p in eq.s (1) & (2) we have

$$x_0 = \frac{\iint x(p_0 + ax)\, dy \cdot dx}{\iint (p_0 + ax)\, dy \cdot dx.} = \frac{\iint ax^2 \cdot dy\, dx}{\iint p_0\, dy\, dx}$$

(Since the expression $\iint x\,dx\,dy$ becomes = zero because the axis Y passes thro. the C. of G. of the surface.)

Placing $\iint x^2 dx\, dy = I$ & remembering that $\iint p_0\, dx\, dy = p_0 \iint dx\, dy = p_0 S = P$, we have

$$x_0 = \frac{aI}{P} \text{ - - - - - - - - - (3)}$$

Similarly

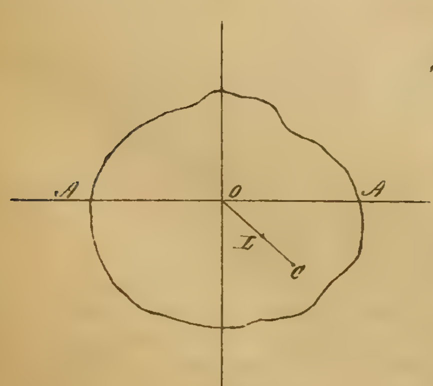
$$y_0 = \frac{a\iint xy\, dx\, dy}{P} = \frac{aK}{P} \text{ - - - - - - (4)}$$

by placing $\iint x \cdot y \cdot dx\, dy = K$.

In Fig. 3.

$$\cot.\, \theta = \frac{y_0}{x_0} = \frac{K}{I}$$

The line CO is called the axis conjugate to OY because, as will afterwards appear, these lines are analogous to the conj. diam.s of an ellipse.

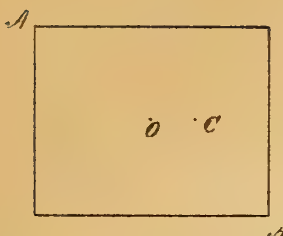


A uniformly varying stress being made up of a uniform stress = P & a uniformly varying stress whose resultant is only = M & this mom.t & force acting in the same or parallel planes the total resultant should be (as it is) a force = P acting at a distance from O (the C. of G. of AA)

$= I = \frac{M}{P} = \sqrt{x_0^2 + y_0^2} = \frac{a\sqrt{I^2 + K^2}}{P}$ which corresponds with article 92.

Cor. to preceding.
(See Rank. C. E. Art. 157.)

— " —



If a certain pressure P be so applied to a symmetrical surface AA as to produce a uniformly varying pressure then the max. intensity of the pressure upon AA will be as much greater than it would be if P were uniformly distributed as the max. value of p $(= p_0 + ax)$ exceeds p_0.

In $p = p_0 + ax$ the max. value occurs when x is a max. or at that point of AA which is farthest from OY. Call this value x_1. Then

$$\text{max. } p = p_1 = p_0 + ax_1$$

But from (3)

$$a = \frac{x_0 P}{I} = \frac{x_0 p_0 S}{I}$$

$$\therefore \quad p_1 = p_0 + \frac{p_0 x_0 S x_1}{I} = p_0\left(1 + \frac{x_0 x_1 S}{I}\right)$$

$$\therefore \quad \frac{p_0}{p_1} = \frac{1}{1 + \frac{x_0 x_1 S}{I}}$$

Now, if we limit this ratio by the consideration that the stress on AA shall be nowhere negative, the value of p at the point of AA farthest to the left of O should be not less than zero, $(p = p_0 - ax_1 \geqq 0 \therefore p_0 \text{ must be } \geqq ax_1)$ & the max. value p_1 will not exceed $2p_0$. In this case

$$\frac{p_0}{p_1} \leqq \frac{1}{2} \leqq \frac{1}{1 + \frac{x_0 x_1 S}{I}}$$

To fulfil this condition, $\frac{x_0 x_1 S}{I} \leqq 1$ or $x_0 \leqq \frac{I}{x_1 S}$

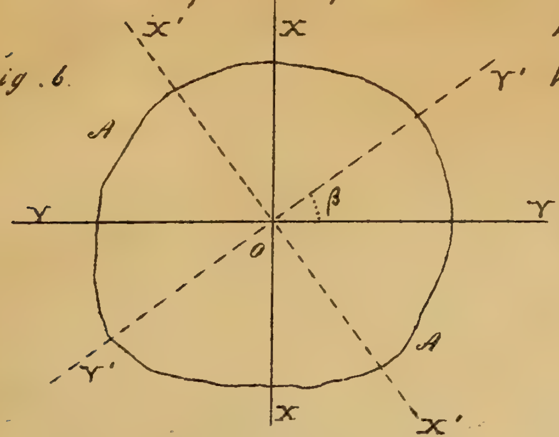
— " —

A. M. Art. 95.
Moments of Inertia — Ellipse of Inertia.

Take any surface AA (Fig. 6) & assume any two rect. axes passing thro. its centre of grav. then the mom. of inertia around $OY = \iint x^2\,dx\,dy = I$, that around $XOX = \iint y^2\,dx\,dy = J$. Place $\iint x.y.\,dx\,dy = K$.

To find the mom.ts of inertia for any other two rect. axes passing thro. O & making an angle $= \beta$ with the first. With reference to these new axes, we have

Fig. 6.



$$I' = \iint x'^2 dx' dy' \; ; \; J' = \iint y'^2 dx' dy'$$
$$K' = \iint x'y' dx' dy'$$

But since

$$x' = x\cos\beta - y\sin\beta \quad \text{and}$$
$$y' = x\sin\beta + y\cos\beta \quad \text{and}$$
$$x'^2 + y'^2 = x^2 + y^2$$

& since $dx'dy' = dx\,dy$ (as each is the differential of the surface), we have

$$I' = I\cos^2\beta + J\sin^2\beta - 2K\cos\beta\sin\beta \text{ ------ } (1)$$
$$J' = I\sin^2\beta + J\cos^2\beta + 2K\cos\beta\sin\beta \text{ ------ } (2)$$
$$K' = (I-J)\cos\beta\sin\beta + K(\cos^2\beta - \sin^2\beta) \text{ ------ } (3)$$

Add (1) & (2) and

$$I + J = I' + J' = \iint (x^2+y^2)\,dx\,dy \text{ ---------- } (4)$$

As this eq. is independent of β, it is "isotropic" or constant; or

Theorem. I. "The sum of the mom.ts of inertia of a surface relatively to any pair of rect. neutral axes passing thro. the C. of G. is constant".

The sum $(I+J) = \iint (x^2+y^2)\,dx\,dy$ is called the "polar moment of inertia". It is the mom.t of inertia with reference to an axis passing thro. O & perp. to the surface AA.

Again, multiply (1) & (2) together, & square (3) & subtract from the result. Then

$$I'J' - K'^2 = IJ - K^2 \text{ ----------- } (5)$$

another constant or isotropic function.

— " —

To prove theorem II of the text.

The sum $(I'+J')$ being constant, whenever I' is a max., J' must be a min. & consequently the difference of the two $(I'-J')$ must then be a max. & when $(I'-J')$ is

a max. its square is also a max. But

$$(I'-J')^2 = (I'+J')^2 - 4I'J' = \text{a max.} \text{ - - - - - } (6)$$

Now the first part $(I'+J')^2$ of this last expression being constant, the expression $I'J'$ must be a min. to make $[(I'+J')^2 - 4I'J']$ a max. Now, $I'J'$ cannot be zero; but from (5) $(I'J' - K'^2)$ = a constant & as K' may be zero, of course $K' = 0$ is that value of K' which makes $I'J'$ a min. Hence $K' = 0$ makes $I'J'$ a min. in (5) & $I'J'$ a min. in (6) makes $(I'-J')$ a max. which in turn gives I' = a max. & J' = a min.

Let I_1 & J_1 be the max. & min. mom.^ts of inertia. The axes about which they are taken are called the Principal Axes. Since $K_1 = 0$, it is easy to find the angle β_1 made by the principal axes with the axes originally assumed. For, from (3) we have

$$(I-J)\cos\beta_1 \sin\beta_1 + K(\cos^2\beta_1 - \sin^2\beta_1) = 0 = K_1$$

$$\therefore -\frac{2K}{I-J} = \frac{2\cos\beta_1\sin\beta_1}{\cos^2\beta_1 - \sin^2\beta_1} = \frac{2\frac{\sin\beta_1}{\cos\beta_1}}{1 - \frac{\sin^2\beta_1}{\cos^2\beta_1}} = \frac{2\tan\beta_1}{1-\tan^2\beta_1} = \tan 2\beta_1 \text{ - - } (7)$$

This eq. gives the position of the principal axes. The values of J_1 and I_1 can be readily obtained from the isotropic functions

$$I_1 + J_1 = I + J \quad \text{and} \quad I_1 J_1 = IJ - K^2$$

from which

$$I_1 = \frac{I+J}{2} + \sqrt{\frac{(I-J)^2}{4} + K^2} \quad ; \quad J_1 = \frac{I+J}{2} - \sqrt{\frac{(I-J)^2}{4} + K^2}$$

Having thus determined the position of the principal axes & the values of I_1 & J_1 we may assume these axes as the original ones & find simpler expressions for the values of I' & J' (the mom.^ts of inert. with reference to any other two rect. axes). Thus eq.^s (1) & (2) & (3) become

$$I' = I_1\cos^2\beta + J_1\sin^2\beta \text{ - - - - - - - - } (8)$$

$$J' = I_1\sin^2\beta + J_1\cos^2\beta \text{ - - - - - - - - } (9)$$

$$K' = (I_1 - J_1)\cos\beta\sin\beta \text{ - - - - - - - - } (10)$$

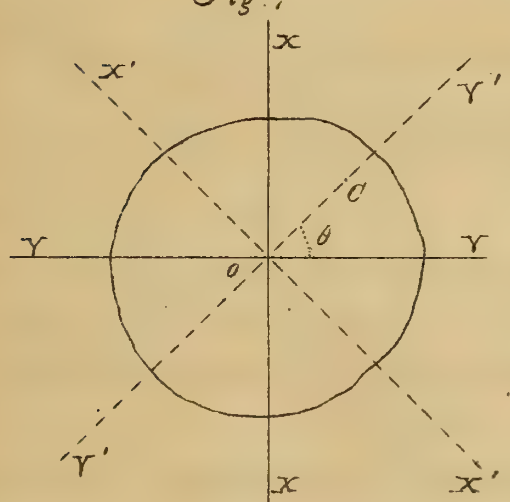
We see from the above that, if $I_1 = J_1$, then $I' = J' = I_1 = J_1$ & $K' = 0$ for all axes whatsoever. This is the case when

the surface AA is a circle.

Conjugate axes. We saw in the last article (94) that the line drawn from the centre of pressure of a surface acted on by a uniformly varying force made with the neutral axis OY an angle whose cotang. $= \frac{K}{I}$ (I being the mom.^{t} of inert. about OY). We will now show how closely the mom.^{ts} of inert. of a surface are related to the centres of pressure of the diff. uniformly varying stresses that may act on that surface. Thus when $K = 0$, $\cot. \theta = 0$ or $\theta = 90°$ i.e.

Theorem. III. The principal axes of a surface are conj. to each other. or - If a surface be acted on by a uniformly varying stress of such kind that the line parallel to AA in the upper surface BB of the cylinder representing the stress (Fig. 3! Art. 94) is also parallel to one of the principal axes of the surface AA: Then the centre of pressure will be on the other principal axis. Again we may prove generally -

Fig. 7



Theorem. IV. That, if the axis conj. to a given neutral axis be taken as a new neut. axis, the original neut. axis will be the new conj. axis. Let us suppose, for instance, that YOY & XOX (Fig. 7.) are the originally assumed axes, & that the new axes to which the mom.^{ts} of inert. are to be referred ($Y'OY'$ & $X'OX'$) make with the original axes an angle ($\beta = \theta$ or, in other words, that $Y'OY'$ passes thro. the centre of pressure. Now

$$\cot. \theta = \frac{K}{I}$$

$$\therefore \quad \cos \theta = \frac{K}{\sqrt{I^2 + K^2}} \quad ; \quad \sin \theta = \frac{I}{\sqrt{I^2+K^2}}$$

If we substitute these values of $\cos\theta$ & $\sin\theta$ for

$\cos\beta$ & $\sin\beta$ in eq.s (1) & (2), we have

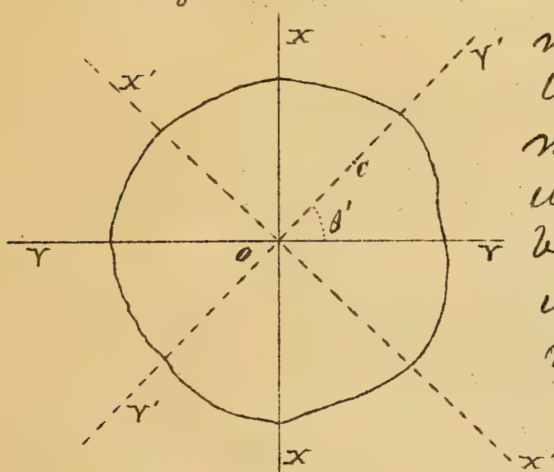
$$I' = \frac{IK^2 + JI^2 - 2IK^2}{I^2+K^2} = \frac{I(IJ-K^2)}{I^2+K^2} \text{ - - - - - } (11)$$

$$K' = \frac{(I-J)IK + K(K^2-I^2)}{I^2+K^2} = -\frac{K(IJ-K^2)}{I^2+K^2} \text{ - - - - } (12)$$

These being the values of I' & K' when the new axis $Y'OY'$ is conj. to the original axis YOY.

Now let us assume (Fig. 8.) $Y'OY'$ & $X'OX'$ as the original axes & let us find the mom.ts of inert. about axes conj. to these. Let θ' = the angle made by the axis conjugate to $Y'OY'$ with $Y'OY'$. Then $\cot\theta$ must be $= \frac{K'}{I'}$. Substituting the values of K' & I' from (11) & (12) we have—

Fig. 8

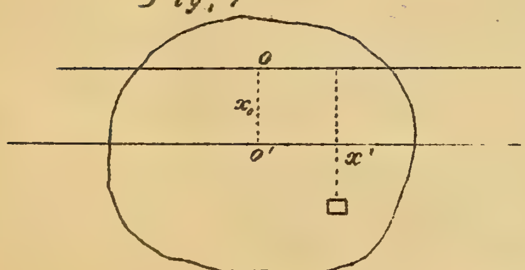


$$\cot\theta' = \frac{K'}{I'} = -\frac{K}{I}$$

But
$$\cot\theta = \frac{K}{I}. \text{ Hence}$$

$$\cot\theta' = -\cot\theta \quad \text{or} \quad \theta' = -\theta$$

Fig. 9

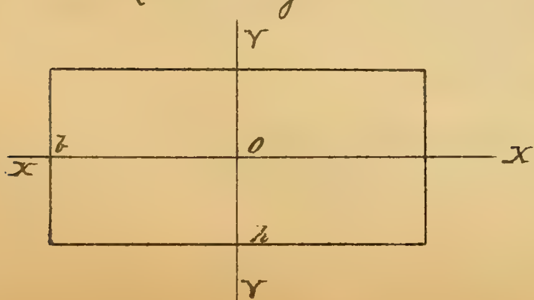


Consequently the axis YOY is conj. to $Y'OY'$, or YOY & $Y'OY'$ are mutually conj. to each other.

Theorem V is simply explained in the text.

Moment of Inertia of a Rectangle

Rectangle



The max. & min. moments in this case are about the axes YOY & XOX in the figure. For, in reference to these axes, $K = \iint xy\,dx\,dy = \frac{y^2}{2}\cdot\frac{x^2}{2}$ which

taken between the limits

$$y = +\frac{b}{2} \text{ \& } y = -\frac{b}{2}, \quad x = +\frac{h}{2} \text{ \& } x = -\frac{h}{2}$$

gives $K = 0$. Moment of inertia about YOY is

$$I_1 = \iint x^2 dx\, dy = \frac{b}{2}\int x^2 dx + \frac{b}{2}\int x^2 dx$$

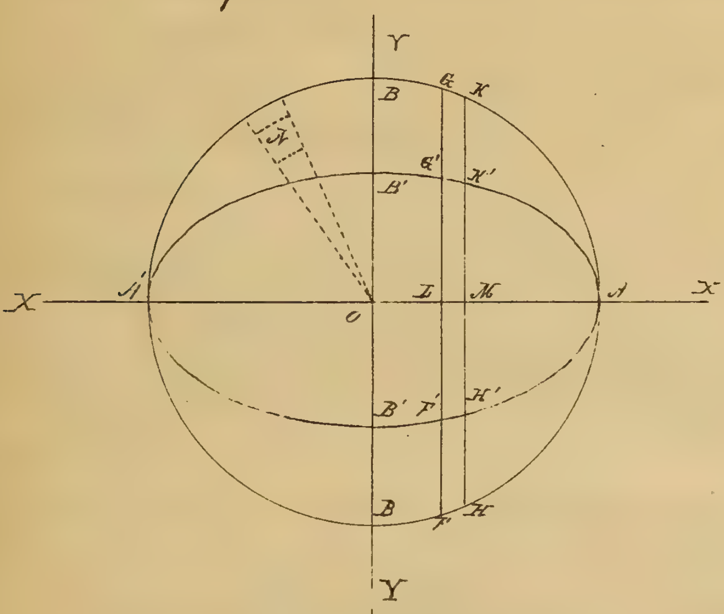
$$= \frac{bx^3}{3} = \frac{bh^3}{12}$$

So

$$J_1 = \iint y^2 dx\, dy = \frac{hb^3}{12}$$

— " —

Ellipse & circle. The ordinates FG', MK' of the ellipse are to the corresponding ones FG, MK of the circle as $OB' : OB$ that is $B : A$ (the semi-diam^s. of the ellipse). Hence the area of the differential slices ($2y'dx$ & $2y\,dx$) are in the same ratio, & ∴ the mom^t of inert. I' of the ellipse with reference to YOY & I of the circle ($\iint 2y'x^2 dx$ & $\iint 2yx^2 dx$) are in the same ratio.



or

$$I' = \frac{B}{A} . I$$

Now in the circle, the mom^ts I & J are equal or $I = J = \iint x^2 dx\, dy = \iint y^2 dx\, dy$. Hence

$$2I = \iint (x^2 + y^2)\, dx\, dy.$$

But $(x^2+y^2) = r^2$ & the differential of the surface corresponding to $dx\,dy$ (or N) referred to polar coordinates is $= r\,dr\,d\theta$.

Hence

$$2I = \iint r^3 dr\, d\theta.$$

Limits of θ are $0°$ & 2π; of r they are 0 & A.

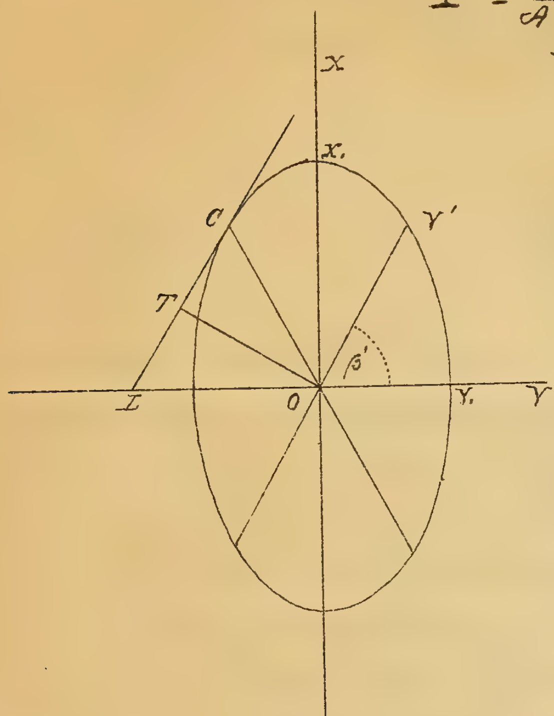
$$\therefore \quad 2I = 2\pi\int r^3 dr = \frac{\pi A^4}{2} \quad \therefore \quad I = \frac{\pi A^4}{4}$$

If $2A = h$ the diam. of the circle, we will have

$$I = \frac{\pi h^4}{64}.$$

In the ellipse
$$I' = \frac{B}{A} \cdot I = \frac{b}{h} \cdot I = \frac{\pi b h^3}{64}.$$

— " —



The relations bet. the mom^ts of inert. of a surface may be represented by an ellipse called the "ellipse of inertia".

Thus lay off the two semi-axes of an ellipse $A = \sqrt{I_1} = OX_1$, & $B = \sqrt{J_1} = OY_1$, & construct an ellipse on them. Then, if OY' be any semi-axis making with OY_1 an angle $= \beta'$, from eq.(k) the mom^t of inert. with reference to OY' is

$$I' = I_1 \cos^2\beta' + J_1 \sin^2\beta'.$$

But, if in the ellipse, we draw OC the diam. conj. to OY', the tang. at C is parallel to OY' & the perp. on that tang. ($= OT$) has for its value

$$\overline{OT}^2 = n^2 = A^2\cos^2\beta' + B^2\sin^2\beta'$$

[For the eq. of the tang. line CI is

$$A^2yy' + B^2xx' = A^2B^2 \quad \text{or} \quad y = -\frac{B^2x'}{A^2y'}x + \frac{B^2}{y'} \text{ ----(1)}$$

& the eq. of OT perp. to this is

$$y = \frac{A^2y'}{B^2x'} \cdot x \text{ ------(2)}$$

Multiply (1) & (2) &

$$y^2 = -x^2 + \frac{A^2}{x'} \cdot x \text{ ----(3)}$$

But from the ellipse

$$A^2y'^2 + B^2x'^2 = A^2B^2 \quad \text{\& from (2)}$$

$$y' = \frac{B^2x'}{A^2x} \cdot y$$

$$\therefore \quad A^2\left[\frac{B^4x'^2y^2}{A^4x^2}\right] + B^2x'^2 = A^2B^2$$

$$\therefore \quad x' = \frac{AB}{\sqrt{\frac{B^4y^2 + B^2x^2A^2}{A^2x^2}}} = \frac{A^2Bx}{\sqrt{B^4y^2 + B^2x^2A^2}}.$$

Substitute this in (3)

$$y^2+x^2 = \frac{\frac{A^2x}{A^2B^4}}{\sqrt{B^4y^2+A^2B^2x^2}} = \sqrt{B^2y^2+A^2x^2}$$

$\therefore (x^2+y^2)^2 = A^2x^2+B^2y^2$ which is the eq. of the locus of I. Pass to polar coordinates, the pole being at O, & we have

$$\left.\begin{aligned} r^4 &= r^2[B^2\sin^2\beta' + A^2\cos^2\beta'] \text{ or} \\ r^2 &= A^2\cos^2\beta' + B^2\sin^2\beta' = n^2 = \overline{O\mathrm{I}}^2 \end{aligned}\right]$$

— " —

But, as $A^2 = I_1$ & $B^2 = J_1$, we have

$$I' = \overline{O\mathrm{I}}^2 = n^2 \text{ or } n = \sqrt{I'}.$$

Also, if $C\mathrm{I} = t$, we will have

$$nt = (A^2-B^2)\cos\beta'\sin\beta' = K'$$

[For $C\mathrm{I} = O\mathrm{I}\tan \mathrm{I}OC = O\mathrm{I}\cot. CO\mathrm{Y}' = O\mathrm{I}\cot(\beta-\beta')$

(calling $CO\mathrm{I}$, β) $\therefore$

$$nt = n^2\cot.(\beta-\beta') = n^2\left[\frac{1+\tan\beta\cdot\tan\beta'}{\tan\beta-\tan\beta'}\right].$$

But from the conj. diam.^{s} of the ellipse

$$B^2\tan\beta\tan\beta' = -A^2 \text{ or } \tan\beta = \frac{-A^2}{B^2\tan\beta'}$$

Substitute

$$nt = n^2\left[\frac{1-\frac{A^2}{B^2}}{-\frac{A^2}{B^2\tan\beta'}-\tan\beta'}\right] = n^2\left[\frac{(A^2-B^2)\tan\beta'}{A^2+B^2\tan^2\beta'}\right]$$

$$= \frac{n^2(A^2-B^2)\sin\beta'\cos\beta'}{A^2\cos^2\beta'+B^2\sin^2\beta'} = (A^2-B^2)\sin\beta'\cos\beta' \quad]$$

Again $\cot\theta = \frac{K'}{I'} = \frac{t}{n} = \cot \mathrm{Y}'OC.$

$$\therefore \theta = \mathrm{Y}'OC.$$

or the angle made by the axis conj. to the neut. axis $O\mathrm{Y}'$ is = that made by the diam. of the ellipse OC with its conj. $O\mathrm{Y}$.

Hence

1° If any diam. of the ellipse be assumed as the neut. axis, the diam. conj. to it will be the conj. axis & these conj. diam.^{s} give the directions of the mutually conjugate axes.

2° If any diam. be assumed as the neut. axis & the tang. to the ellipse be drawn thro. the extremity of the conj. diam. then the square of the perp. from the centre on that tang. is = the mom.^t of inert. about the assumed neutral axis on the same scale on which the square of the principal semi-axes represent the max. & min. mom.^ts of inert.

— " —

Ellipse of Stress.

A. M. Art. 112.

Fig. 1

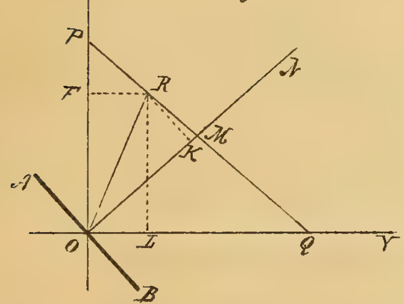


Fig. 2

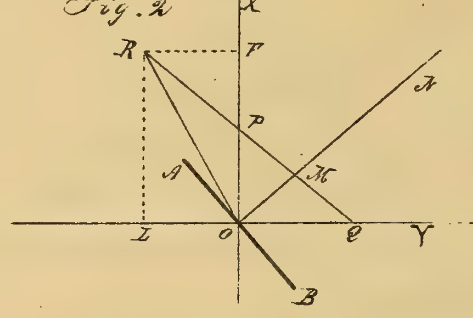
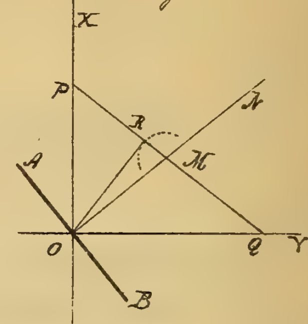


Fig. 3



Prob. I. Since $OM = MQ = \frac{p_x + p_y}{2}$ and $MR = \frac{p_x - p_y}{2}$ by adding & subtracting we have $RQ = p_x$ & $PR = p_y$.

For convenience let us put $p_x = A$ & $p_y = RP = B$. The coordinates of R are $-OL = y$ & $LR = x$. The resultant $OR = p_r$ is expressed by

$$p_r^2 = (x^2+y^2) = \overline{RQ}^2 \cos^2 x\hat{n} + \overline{PR}^2 \sin^2 x\hat{n} =$$
$$A^2 \cos^2 x\hat{n} + B^2 \sin^2 x\hat{n}.$$

(Since $\angle MQO = 90° - x\hat{n}$)

In triangle RMO

$$\sin RMO : \sin \hat{nr} :: OR : RM$$

But $\angle RMO = 180° - 2x\hat{n}$ ∴ $\sin RMO = \sin 2x\hat{n}$

$$\therefore \sin \hat{nr} = \sin 2x\hat{n} . \left(\frac{A-B}{2p_r}\right)$$

Again, in the triangle RLQ

$$x^2 = A^2 - A^2 \sin^2 x\hat{n}$$

& in the triangle RFP

$$B \sin x\hat{n} = FR = y$$

$$\therefore \sin . x\hat{n} = \frac{y}{B}$$

Substitute, &

$$A^2y^2 + B^2x^2 = A^2B^2 \text{ (eq. of an ellipse)}.$$

Drop the perp. RK from R upon ON; then the normal & tangential components of OR are

$OR \cos \hat{nr} =$ (by substituting & reducing) $A \cos^2 \hat{xn} + B \sin^2 \hat{xn}$.

$OR \sin \hat{nr} =$ (" " " ") $(A-B) \cos \hat{xn} . \sin \hat{xn}$.

– " –

Prob II – The shearing or tangential stress is evidently a max. when $RM (= \frac{A-B}{2})$ is itself the perp.; in which case, since the angle $PMO = 90°$, each of the angles $MOP (= \hat{xn})$ & OPM must $= 45°$. This result is also attained by applying the tests for a max. to the expression above for the tangential force.

– " –

Prob. III. As MR is constant as well as OM, it is evident that the angle MOR will be a max. when OR is so situated as to be tang. to the arc that may be described about M with a radius = MR. When this is the case, MR is perp. to OR or the angle MRO is a right angle. The values in this case for $\hat{xn}$, $\hat{nr}$, OR are easily obtained. The last may be gotten geometrically by considering that, in the right-angled triangle POQ, OR is a perp. on the base PQ & ∴

$$OR = \sqrt{PR . RQ} = \sqrt{A . B}$$

Fig. 4

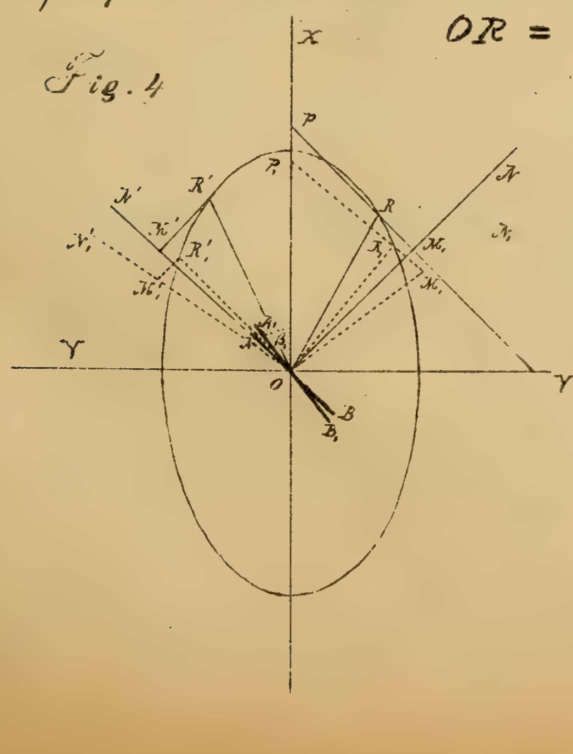


Fig (4) illustrates Prob.s II & III. The full lines apply to Prob. II & the dotted ones to Prob. III. The eq. of OA & OB is

$$y = \tan. 45° . x \;;$$

that of the lines OA_1 & OB_1 is also of the form

$$y = ax \qquad \text{where}$$

$a =$ the tang. of the angle XOA_1. Call this β_1 & then from eq. 8. page 105

$$\beta_1 = \frac{90^\circ - \sin^{-1}\left(\frac{A-B}{A+B}\right)}{2}$$

$$\therefore\ 90^\circ - 2\beta_1 = \sin^{-1}\left(\frac{A-B}{A+B}\right) = \tan^{-1}\left(\frac{A-B}{2\sqrt{A.B}}\right)$$

$$\therefore\ \frac{A-B}{2\sqrt{A.B}} = \tan(90^\circ - 2\beta_1) = \cot 2\beta_1 = \frac{1-\tan^2\beta_1}{2\tan\beta_1}$$

$$\therefore\ \tan\beta_1 = \frac{B}{\sqrt{A.B}}$$

Hence, the eq. of the line is

$$y = \frac{B}{\sqrt{A.B}}\cdot x\ .$$

Fig. 5

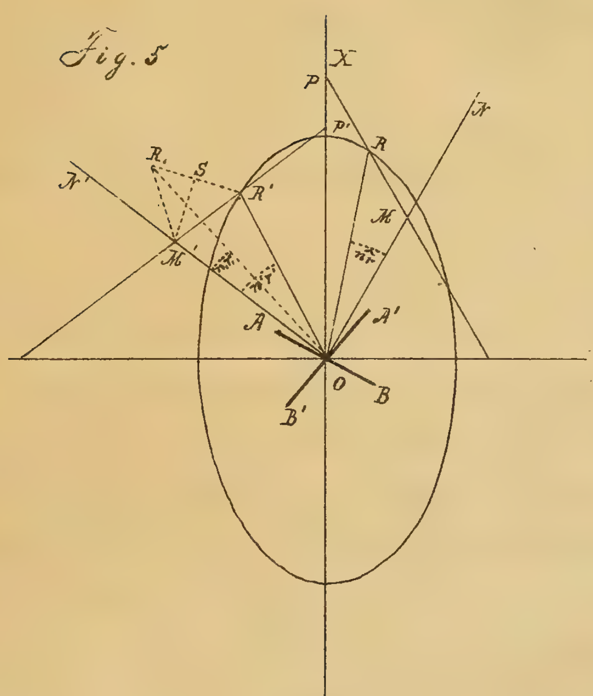


Fig. 5. illustrates Prob. IV when the given stresses are of the same kind & unequal. Here we have given the two stresses OR & OR' & the angles $\hat{nr}$ & $\hat{n'r'}$ to determine the "principal axes" of the ellipse of stress -

1° To find the compts. OM = OM' & MR = MR' of the stresses OR & OR'

Geometrically.

Conceive the angle NOR to be turned over & laid upon N'OR' placing ON upon ON'. The line OR will fall at OR_1 (on the same side of ON' with OR' when the stresses are of the same kind). Draw $R'R_1$ & at S the middle point erect the perp SM'. From M' draw $M'R_1$. Then OM' = OM & M'R' = MR are evidently the required compts.; one of which = the half-sum & the other the half difference of the axes of the ellipse. From

$$OM = \frac{A+B}{2} = OM' \quad\text{and}\quad MR = \frac{A-B}{2} = M'R'$$

add & subtract to find A. & B.

Having found the point M, we know that the angle NMR = $2\,\hat{xn}$ or $\hat{xn} = \frac{1}{2}$ NMR which fixes the position of the axis XX.

The angle $\hat{nn'}$ bet. the normals = $\hat{xn} + \hat{xn'}$

$$= \frac{N'M'R_1 + N'M'R'}{2} = N'M'S.$$

Algebraically

From the two triangles OMR & $OM'R'$ we have

$$R\bar{M}^2 = O\bar{M}^2 + O\bar{R}^2 - 2\,O\bar{M}.O\bar{R}.\cos \hat{n}r$$
$$= R'\bar{M}'^2 = O\bar{M}^2 + O\bar{R}'^2 - 2\,O\bar{M}.O\bar{R}'\cos \hat{n'}r'$$

$$\therefore \quad OM = \frac{A+B}{2} = \frac{p^2 - p'^2}{2(p\cos \hat{n}r - p'\cos \hat{n'}r')}$$

$$\& \quad MR = M'R' = \sqrt{\left(\frac{A+B}{2}\right)^2 + p^2 - (A+B)p.\cos \hat{n}r}$$

and

$$\cos NMR = \cos 2\angle n = \frac{p\cos \hat{n}r - \frac{A+B}{2}}{\frac{A-B}{2}}$$

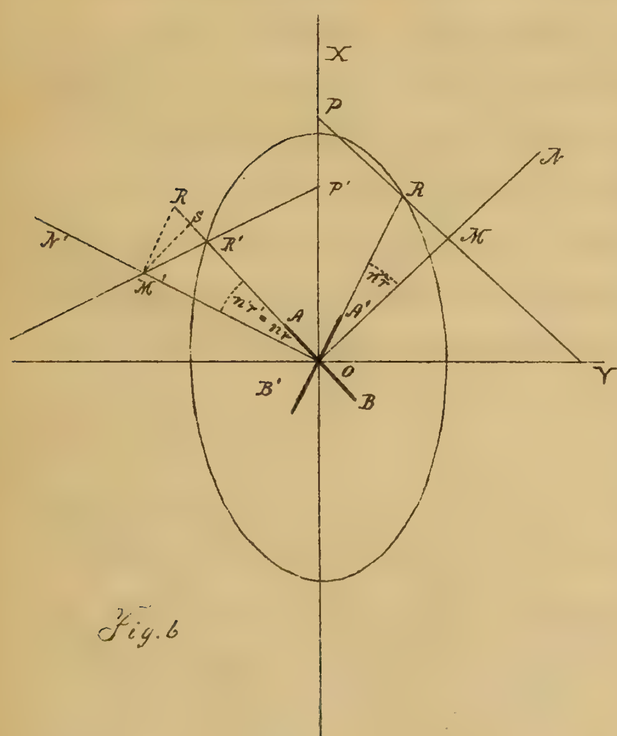


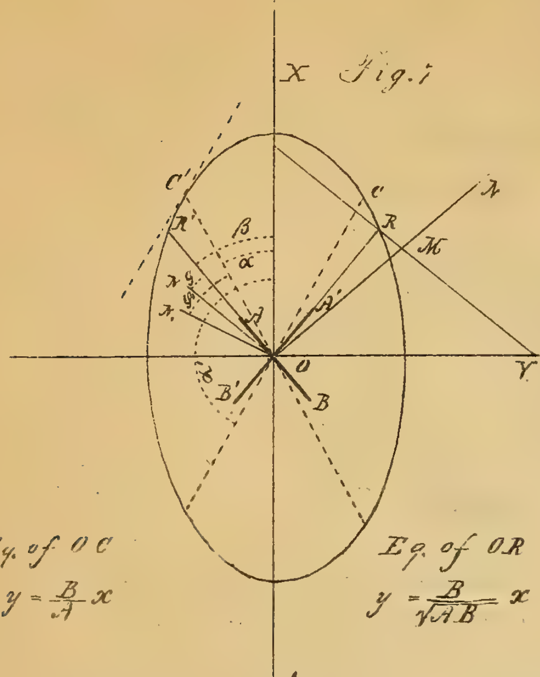
Fig. 6

Fig. 6 illustrates Case 3 Prob. IV where the given stresses are conj. The obliquities being equal in this case, $\hat{n}r = \hat{n'}r'$ & when $\angle NOR$ is placed upon $\angle N'OR'$, OR falls upon OR' & the triangles MOR & $M'OR'$ assume the positions shown in the figure. The semi-diam^s. of the ellipse representing "conj. stresses" (such as OR & OR' Fig. 6) must not be confounded with what are known as the conj. diam^s of the ellipse. i.e. pairs of diameters, each parallel to the tang.^s drawn thro. the extremities of the other. The lines representing the conj. stresses have many useful properties & relations to the semi-axes, some of which are given in eq^s (19) & (20); & others may be gotten from these. Thus, if the conj. stresses (p & p') are put $= A'' + B''$, by squaring eq^s (19) & (20) & adding, we obtain

$$A''B'' = AB \text{ - - - - - - - - - - - - } (1)$$

&, dividing (19) by (20) & squaring, we obtain

$$\left(\frac{A-B}{A+B}\right)^2 = 1 - \frac{4A''B''\cos^2 \widehat{nr}}{(A''+B'')^2} \quad \text{-----------} (2)$$



E. The relation used in Prob. V for finding the ratio $A'' : B''$.

In the case of the conj. diam.s (A' & B'), it will be remembered that $A'B' \sin(\alpha' - \alpha) = AB$ where $(\alpha' - \alpha)$ = angle bet. the diam.s. The diam.s representing the conj. ~~di-~~ ~~ams~~ stresses of max. obliquity are seen from Prob. III to be equal, and, consequently, each $= \sqrt{AB}$. So too the conj. diam's of max. obliquity are equal. These two sets of lines are shown in Fig. 7. The dotted lines are the conj. diam.s. The lines representing the conj. stresses of max. obliquity are there seen to fall outside of the conj. diam.s of max. obliquity. The principal axes belong to both these sets of lines.

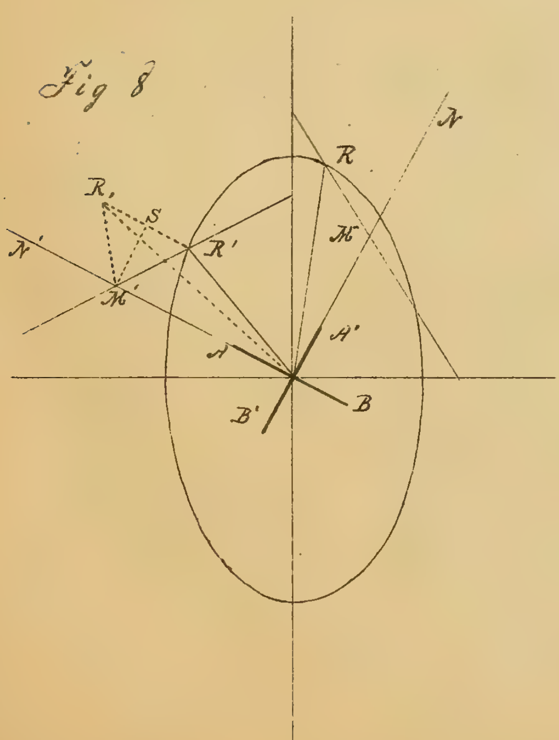


Fig. 8. illustrates Case 4 Prob. IV. & $\widehat{nn'}$ in this case = 90° &, since $\widehat{nn'} = N'M'S$, this latter angle must = 90°. Hence when we lay NOR on $N'OR'$ & complete the figure as before, we will find RR_1 parallel to ON' & SM' perp. to both. The normal comp.ts

$$p_n^2 = p^2\cos^2\widehat{nr} = p^2(1-\sin^2\widehat{nr}) \quad \&$$

$$p_n'^2 = p'^2\cos^2\widehat{n'r'} = p'^2(1-\sin^2\widehat{n'r'})$$

$$\therefore p_n^2 - p_n'^2 = p^2 - p'^2$$

Hence we readily get eq.s (21), (22), & (23). The object of Prob. V is to express the ratio of any two diam.s representing conjugate stresses in terms of the obliquity of

the stresses & of the max. obliquity of stress in the ellipse to which they belong. The ratio of the stresses in terms of the principal axes & their own obliquity may be gotten from expression (2) above, but it is often more convenient to have this ratio in the form given in Prob. V.

For eq.s (25) & (26), see any discussion of eq.s of the second degree. Write eq. (25) thus

$$\frac{(p+p')^2}{pp'} = 4\frac{\cos^2 \hat{nr}}{\cos^2\varphi}$$

Then, if p & p' be considered as roots of a quadratic eq., the first member of this eq. is the ratio of the square of the sum of these roots to their product, & the last member must give the ratio bet. the square of the coefficient of the first power of the unknown quantity & the absolute term of the eq. Hence eq. (26) – When the obliquity $\hat{nr} = 0$, the lines representing the conjugate stresses are the semi-axes, & their ratio is given in eq. (28) in terms of φ.

When $\hat{nr} = \varphi$, the ratio $= 1$, as it should be; as in that case, the conj. stresses are represented by OR & OR' (Fig. 7) which are equal. The relation that exists between the max. obliquity of the conjugate diameters of the ellipse & the max. obliquity of the conj. stresses may be found thus –

In the case of the conj. diam.s which are most oblique to each other, we have

$$\tan\alpha . \tan\alpha' = -\frac{B^2}{A^2} = -\tan^2\alpha$$

(since in the case of these conj. diam.s, $(\alpha'+\alpha) = 180°$) Calling the obliquity of these diam.s φ', we have

$$\alpha = \frac{90° - \varphi'}{2}. \quad \text{Hence}$$

$$\frac{B^2}{A^2} = \tan^2\left(\frac{90°-\varphi'}{2}\right) = \frac{1-\sin\varphi'}{1+\sin\varphi'} \quad \therefore \sqrt{\frac{1-\sin\varphi'}{1+\sin\varphi'}} = \frac{B}{A}$$

But from eq. (28) page 109, $\frac{1-\sin\varphi}{1+\sin\varphi} = \frac{B}{A}$. If we put

$\frac{1-\sin\varphi}{1+\sin\varphi} = \sqrt{\frac{1-\sin\varphi'}{1+\sin\varphi'}}$, we find $\frac{2\sin\varphi}{1+\sin^2\varphi} = \sin\varphi'$.

—"—

Suspension Bridge.

Rank. C. E. Art. 125.

Taking the eq. $\frac{dy}{dx} = \frac{P}{H}$ & substituting px for P, we have $\frac{dy}{dx} = \frac{px}{H}$ $\therefore$ $y = \frac{p}{2H}\cdot x^2$ $\therefore$ $x^2 = \frac{2H}{p}\cdot y$ ------(1)

which is the eq. of a parabola with origin at A & axis vert. If the focal distance of this parabola $= m$, then its eq. may be written

$$x^2 = 4my. \text{ ----------(2)}$$

where $m = \frac{H}{2p} = \frac{x^2}{4y}$. Also

$$\tan i = \frac{dy}{dx} = \frac{x}{2m} = \frac{2y}{x} \text{ and } \sec i = \sqrt{1+\frac{dy^2}{dx^2}} = \sqrt{1+\frac{x^2}{4m}} = \sqrt{1+\frac{4y^2}{x^2}} \text{ --(3)}$$

—..—

Prob. I For the two piers, the eq.s become

$x_1^2 = 4my_1$ and $x_2^2 = 4my_2$ whence

$\sqrt{y_1} : \sqrt{y_2} :: x_1 : x_2$ and by composition

$$x_1 = \frac{a\sqrt{y_1}}{\sqrt{y_1}+\sqrt{y_2}}, \quad x_2 = \frac{a\sqrt{y_2}}{\sqrt{y_1}+\sqrt{y_2}} \text{ ------ (4)}$$

$$x_1 + x_2 = a$$

Again $x_1^2 : 4y_1 :: x_2^2 : 4y_2$ by alternation & composition $x_1^2 + x_2^2 : x_2^2 :: 4y_1 + 4y_2 : 4y^2$

$$\therefore \frac{x_2^2}{4y_2} = \frac{x_1^2+x_2^2}{4(y_1+y_2)} = m = \frac{x_1^2}{4y_1}$$

Substituting for x_1 & x_2 from (4), we have

$$\left.\begin{aligned} m = \frac{x_1^2+x_2^2}{4(y_1+y_2)} &= \frac{a^2y_1 + a^2y_2}{y_1 + 2\sqrt{y_1y_2} + y^2}\cdot\frac{1}{4(y_1+y_2)} \\ &= \frac{a^2}{4(y_1+y_2)+8\sqrt{y_1y_2}} \end{aligned}\right\} \text{ ---- (5)}$$

Prob. II. In $\tan i_1 = 2\frac{y_1}{x_1}$ substitute value of x_1 above.

Prob. IV. The eq. expressing the length of a parabolic arc from the origin A to a point whose ordinate is x is given in Courtenay's Calculus. p. 324. Taking that eq. (substituting for p its value $2m$ & for $\frac{b}{2}$ its value x) we have

$$s = \frac{x\sqrt{4m^2+x^2}}{4m} + m.\ \text{nap. log.}\left[\frac{\sqrt{4m^2+x^2}+x}{2m}\right] \text{-----}(9)$$

Substitute for m its value $\frac{x^2}{4y}$ & we have

$$s = \sqrt{y^2+\frac{x^2}{4}} + \frac{x^2}{4y}\cdot \text{nap. log.}\left[\frac{y+\sqrt{y^2+\frac{1}{4}x^2}}{\frac{x}{2}}\right] \text{------}(10)$$

This is the value of s in terms of its coordinates x & y.
To get another value, take (9) & reduce as follows –

$$\frac{x\sqrt{4m^2+x^2}}{4m} = \frac{2mx\sqrt{1+\frac{x^2}{4m^2}}}{4m} = \frac{x}{2}\sqrt{1+\frac{x^2}{4m^2}} = m.\frac{x}{2m}\sqrt{1+\frac{x^2}{4m^2}}$$

But $\frac{x}{2m} = \tan i$, & $\sqrt{1+\frac{x^2}{4m^2}} = \sec i$. Hence the last expression equals

$$m(\tan i.\sec i)$$

Again

$$\frac{\sqrt{4m^2+x^2}+x}{2m} = \frac{\sqrt{4m^2+x^2}}{2m}+\frac{x}{2m} = \sqrt{1+\frac{x^2}{4m^2}}+\frac{x}{2m} = \sec i + \tan i.$$

Hence (9) may be written

$$s = m\left[\tan i.\sec i + \text{nap. log.}(\tan i + \sec i)\right] \text{-------}(11)$$

Instead of these exact formulae, we may use that for the length of an arc of the osculatory circle at the vertex A of the parabola. This formula is

$$s' = r\left[\frac{x}{r} + \frac{1}{2.3}\cdot\frac{x^3}{r^3} + \frac{1.3}{2.4.5}\cdot\frac{x^5}{r^5} + \text{\&c}\right]$$

& this formula will be unchanged when the origin is transf~~orred~~ from the centre of the circle to the circumference, provided the axis of y is still unchanged; for in that case the ordinates x are the same. The radius of the circle which is osculatory to the parabola at the vertex $= 2m$.

$\therefore\ r = 2m = \frac{x^2}{2y}$. Substitute in the series &

$$s' = x + \frac{2}{3}\cdot\frac{y^2}{x} + x = s \text{ nearly.}$$

C.E. Art. 126.

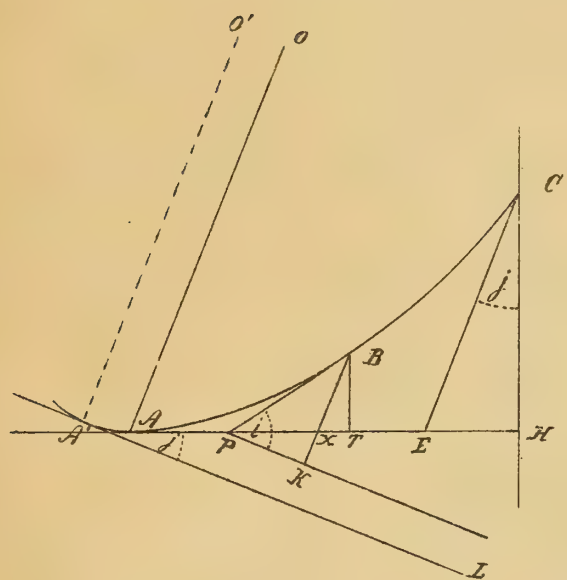
The eq. of a parabola referred to oblique co-ordinates – the tang. at A (not the vertex) & a diam. – with origin at A on the curve is

$$x^2\cos^2 j = 4my. \text{ whence } m = \frac{x^2\cos^2 j}{4y} \text{ -------- (16)}$$

The triangle BPX gives the length of the tang. BP = t as follows –

$$BP = t = \sqrt{\frac{x^2}{4} + y^2 + xy\sin j} \text{ ---------- (17)}$$

Problems similar to I & II of the preceding case are not needed here. To obtain (similarly to Prob. II of that case) the relations bet. the forces acting on a suspension bridge with inclined rods: Let v now = the intensity of the vert. load per unit of length of the hor. platform, per foot for example. Let p = intensity of oblique load & q = rate at which the thrust along the platform increases from A towards H.



Then

$$V = vx\ ;\ P = px = vx\sec j\ ;\ Q = qx = vx\tan j.$$

$$H = \frac{xP}{2y} = \frac{px^2}{2y} = \frac{2pm}{\cos^2 j} = 2vm\sec^3 j$$

$$R = \frac{tP}{y} = \frac{2tH}{x} = \frac{ptx}{y}.$$

To find the length of the parabolic arc AB.

Let i denote the inclination of the parabola at the point B to a line perp. to its axis. Thus, if PK is perp. to BX, the angle BPK = i & XPK = j. Now

$$\cos KPB = \frac{PK}{PB} = \frac{PX\cdot\cos j}{t} = \frac{x}{2t}\cdot\cos j = \cos i$$

$$\therefore\ i = \cos^{-1}\left(\frac{x}{2t}\cdot\cos j\right)$$

When B coincides with A, $i = j$. If we complete the semi-parabola CBA to A', A'O' will be the principal diam. & A' the vertex; A'Z the tang. at the vertex. Then j = the inclination of the curve at A to A'Z & i = inclination of curve at B to same line. Then length of AB

$$s = m\left[\tan i.\sec i - \tan j \sec j + \text{nap. log.}\frac{\tan i + \sec i}{\tan j + \sec j}\right]$$

The approximate formula similar to the one in the preceding case is

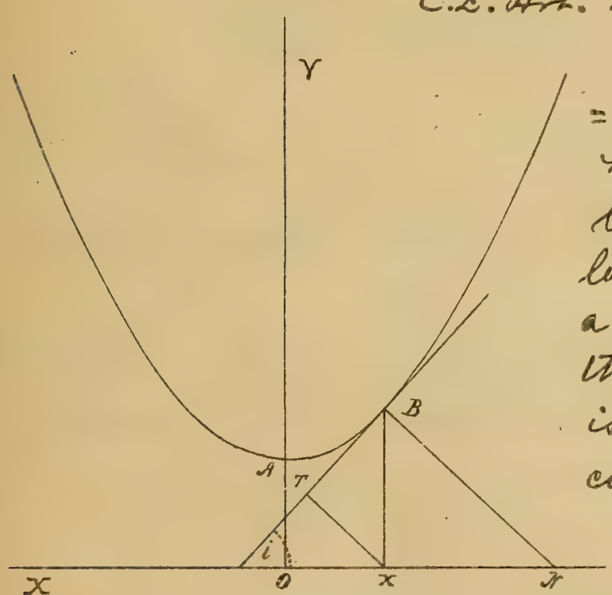
$$s' = x + y\sin j + \tfrac{2}{3}\cdot\frac{y^2\cos^2 j}{x + y\sin j} + xa = s \text{ nearly}.$$

in which AT($= x + y\sin j$) represents the same as x did in the former case.

— " —

Catenary.

C.E. Art. 128.



Let O be the origin & AO $= m =$ modulus of the catenary: Let s = length of any arc: Then by definition, $P = ps$ (where p = load on unit of length of arc & a constant). If i = inclination of the tang. at any point as B to the axis x, then we have

$$\cos i = \frac{dx}{ds}\ ;\ \sin i = \frac{dy}{ds} = \sqrt{1 - \frac{dx^2}{ds^2}} \quad \cdots (1)$$

$$\tan i = \frac{dy}{dx} = \frac{\sqrt{1 - \frac{dx^2}{ds^2}}}{\frac{dx}{ds}} \quad \cdots\cdots (1)$$

The hor. tens.^{n} at A $= H = pm$, if m be the modulus. Then

$$\tan i = \frac{\sqrt{1 - \frac{dx^2}{ds^2}}}{\frac{dx}{ds}} = \frac{P}{H} = \frac{ps}{pm} = \frac{s}{m} \quad \cdots\cdots (3)$$

(Since $\tan i = \frac{P}{H}$ is the general eq. of equilibrium of a cord under vert. loads, see C.E. eq. 2. Art. 123.)

Squaring (3) we have

$$1 - \frac{dx^2}{ds^2} = \frac{s^2}{m^2}\cdot\frac{dx^2}{ds^2} \quad \therefore \quad \frac{dx}{ds} = \frac{m}{\sqrt{m^2+s^2}} \quad \text{or} \quad \frac{dx}{m} = \frac{ds}{\sqrt{m^2+s^2}} \quad \cdots\cdots (4)$$

To integrate this, place $\sqrt{s^2+m^2} = z-s$. Then

$s^2+m^2 = z^2-2zs+s^2 \therefore s = \frac{z^2-m^2}{2z}$ and

$$ds = \frac{2z^2+2m^2}{4z^2} \cdot dz. \qquad \sqrt{s^2+m^2} = z-s$$

$$= z - \frac{z^2-m^2}{2z} = \frac{z^2+m^2}{2z}$$

$$\therefore \frac{ds}{\sqrt{s^2+m^2}} = \frac{dz}{z} \text{ and } \int\frac{ds}{\sqrt{s^2+m^2}} = \int\frac{dz}{z} = \log z + C.$$

$\therefore \frac{x}{m} = C + \log.[s+\sqrt{s^2+m^2}]$. But when $s=0$, $x=0$

& $\therefore$ $$C = -\log. m$$

$$\therefore \frac{x}{m} = \log.[s+\sqrt{s^2+m^2}] - \log. m$$

$$\therefore x = m.\log\left[\frac{s}{m} + \frac{\sqrt{s^2+m^2}}{m}\right] \text{ ---- (Eq. 5. A.M.)}$$

Again $\frac{x}{m} = \log\left[\frac{s}{m}+\frac{\sqrt{s^2+m^2}}{m}\right]$ may be written

$$\varepsilon^{\frac{x}{m}} = \frac{s}{m} + \frac{\sqrt{s^2+m^2}}{m} \text{ or } \varepsilon^{\frac{x}{m}} - \frac{s}{m} = \frac{\sqrt{s^2+m^2}}{m}$$

Squaring $$\varepsilon^{\frac{2x}{m}} - 2\frac{s}{m}\cdot\varepsilon^{\frac{x}{m}} + \frac{s^2}{m^2} = \frac{s^2}{m^2}+1$$

$\therefore \frac{s}{m} = \frac{\varepsilon^{\frac{2x}{m}}-1}{2\varepsilon^{\frac{x}{m}}}$ whence the arc in terms of the abscissa is

$$s = AB = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right] \text{ --- (Eq. 1. C.E. Art. 128)}$$

The ordinate y is found in terms of x by integrating the eq. $\frac{dy}{dx} = \sqrt{\sec^2 i - 1} = \sqrt{\frac{ds^2}{dx^2}-1} = \frac{s}{m} = \frac{1}{2}\left[\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right]$

$$\therefore \quad y = \frac{1}{2}\int(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}})\,dx = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right] + C.$$

When $x=0$, $y=m$; $\therefore C=0$ and

$$y = XB = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right] \text{ ---- (eq. 1. C.E. Art. 128.)}$$

To find y in terms of s. Square the value of s above, add m^2 & we have

$$s^2+m^2 = \frac{m^2}{4}\left[\varepsilon^{\frac{2x}{m}} - 2 + \varepsilon^{-\frac{2x}{m}}\right] + m^2 = \frac{m^2}{4}\left[\varepsilon^{\frac{2x}{m}} + 2 + \varepsilon^{-\frac{2x}{m}}\right]$$

$$= \frac{m^2}{4}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right]^2 \quad \therefore \quad y^2 = s^2+m^2 \text{ (Eq. C.E. Art. 128)}$$

To find the abscissa in terms of the ordinate.

Take the eq.

$$y = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right] \qquad \text{whence}$$

$$2\frac{y}{m} = \varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}} = \varepsilon^{\frac{x}{m}} + \frac{1}{\varepsilon^{\frac{x}{m}}} = \frac{\varepsilon^{2\frac{x}{m}}+1}{\varepsilon^{\frac{x}{m}}}$$

$$\therefore \frac{2y}{m}\cdot\varepsilon^{\frac{x}{m}} = \frac{\varepsilon^{\frac{2x}{m}}+1}{1} \text{ or } 1 - \frac{2y}{m}\cdot\varepsilon^{\frac{x}{m}} = -\varepsilon^{\frac{2x}{m}}$$

adding to both members of this eq. $\left(\frac{y}{m}\right)^2\cdot\varepsilon^{\frac{2x}{m}}$, we have

$$1 - \frac{2y}{m}\cdot\varepsilon^{\frac{x}{m}} + \left(\frac{y}{m}\right)^2\cdot\varepsilon^{\frac{2x}{m}} = \varepsilon^{\frac{2x}{m}}\left[\left(\frac{y}{m}\right)^2 - 1\right]$$

$$\therefore 1 - \frac{y}{m}\,\varepsilon^{\frac{x}{m}} = \varepsilon^{\frac{x}{m}}\sqrt{\left(\frac{y}{m}\right)^2-1}$$

$$\therefore \varepsilon^{\frac{x}{m}} = \frac{1}{\frac{y}{m}+\sqrt{\left(\frac{y}{m}\right)^2-1}} \quad . \quad \therefore \frac{x}{m} = \log.\left[\frac{1}{\frac{y}{m}+\sqrt{\left(\frac{y}{m}\right)^2-1}}\right]$$

The area $AOXB = \int y\,dx = \frac{m}{2}\int\left[\varepsilon^{\frac{x}{m}}+\varepsilon^{-\frac{x}{m}}\right]dx$

$$= \frac{m^2}{2}\left[\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right] \quad (\text{since } C = 0)$$

$$= ms \text{ ----- } (\text{Eq. 1. Art. 128. C.E.})$$

The slope at B

$$\frac{dy}{dx} = \tan i = \frac{s}{m} = \frac{1}{2}\left[\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right] \text{ ... } (\text{Eq. 1. Art. 128. C.E.})$$

To find the radius of curvature.

$$\rho = \frac{\left[1+\frac{dy^2}{dx^2}\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = \frac{s}{m} \quad \therefore \quad \frac{dy^2}{dx^2} = \frac{s^2}{m^2} = \frac{y^2-m^2}{m^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{m}\cdot\frac{ds}{dx} = \frac{1}{2m}\left[\varepsilon^{\frac{x}{m}}+\varepsilon^{-\frac{x}{m}}\right] = \frac{1}{m}\cdot\frac{y}{m} = \frac{y}{m^2}$$

$$\therefore \rho = \frac{\left(\frac{y^2}{m^2}\right)^{3/2}}{\frac{y}{m^2}} = \frac{y^2}{m} = \frac{m}{4}\left[\varepsilon^{\frac{2x}{m}}+\varepsilon^{-\frac{2x}{m}}+2\right] \text{ -- } (\text{Eq. 1. Art. 128. C.E.})$$

$$\text{or} \quad \rho = \frac{y^2}{m} = \frac{m^2+s^2}{m} = m\sec^2 i$$

At A, $\rho = m$ (since $x = 0$ at that point).

If the triangle BTX be constructed with a right

angle at I & the side BI coincident with the tang. at B, then

$$BI = BX \cos IBX = y \sin i$$

But since
$$\sin^2 i = \frac{\tan^2 i}{1+\tan^2 i} = \frac{s^2}{s^2+m^2} \text{ and } y = \sqrt{s^2+m^2}$$

$$\left.\begin{array}{l} BI = \dfrac{\sqrt{s^2+m^2}}{\sqrt{s^2+m^2}} \cdot s = s \text{ and } \\ IX = \sqrt{BX^2 - BI^2} = \sqrt{y^2 - s^2} = m \end{array}\right\} \cdots (\text{Art. } 128. \text{ C.E.})$$

So
$$BN = BX \sec i = y\sqrt{1+\tan^2 i} = y \cdot \frac{\sqrt{s^2+m^2}}{m}$$

$$= \frac{y^2}{m} = \text{radius of curvature.}$$

Again at B
$$R = \sqrt{H^2+P^2} = p\sqrt{m^2+s^2} = py \cdots (\text{Eq. 2. Art. } 128. \text{C.E})$$

From this we see that, if $AO (= m)$ be taken to represent the tens.^{n} on the chain at A, then y will represent the tens.^{n} at any other point. on the same scale, or, in other words, that the "extrados" of the geometrical surface representing the tens.^{n} at various points is a straight hor. line.

— " —

In the Problem on page 197. C.E., let

$$x_1 + x_2 = h \text{ (an unknown quantity)}$$

then
$$x_1 = \frac{h+k}{2} \ \& \ x_2 = \frac{h-k}{2} \quad (\text{since } x_1 - x_2 = k)$$

If we substitute these values in the eq. already found giving s in terms of x, we have

$$s_1 - s_2 = l = \frac{m}{2}\left[\varepsilon^{\frac{h+k}{2m}} - \varepsilon^{-\frac{h+k}{2m}} - \varepsilon^{\frac{h-k}{2m}} + \varepsilon^{-\frac{h-k}{2m}}\right]$$

Factoring
$$s_1 - s_2 = l = \frac{m}{2}\left[\varepsilon^{\frac{h}{2m}} + \varepsilon^{-\frac{h}{2m}}\right]\left[\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right] \cdots\cdots\cdots (12)$$

So too
$$y_1 - y_2 = v = \frac{m}{2}\left[\varepsilon^{\frac{h+k}{2m}} + \varepsilon^{-\frac{h+k}{2m}} - \varepsilon^{\frac{h-k}{2m}} - \varepsilon^{-\frac{h-k}{2m}}\right]$$

Factoring
$$y_1 - y_2 = v = \frac{m}{2}\left[\varepsilon^{\frac{h}{2m}} - \varepsilon^{-\frac{h}{2m}}\right]\left[\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right] \cdots\cdots (12')$$

Squaring (12) & (12')

$$l^2 = \frac{m^2}{4}\left[\varepsilon^{\frac{h}{m}} + 2 + \varepsilon^{-\frac{h}{m}}\right]\left[\varepsilon^{\frac{k}{m}} - 2 + \varepsilon^{-\frac{k}{m}}\right]$$

$$v^2 = \frac{m^2}{4}\left[\varepsilon^{\frac{h}{m}} - 2 + \varepsilon^{-\frac{h}{m}}\right]\left[\varepsilon^{\frac{k}{m}} - 2 + \varepsilon^{-\frac{k}{m}}\right]$$

Subtracting

$$l^2 - v^2 = \frac{m^2}{4}\left[4\left(\varepsilon^{\frac{k}{m}} - 2 + \varepsilon^{-\frac{k}{m}}\right)\right]$$

$$\therefore \sqrt{l^2 - v^2} = m\left[\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right] \cdots\cdots \quad (13) \text{—(Eq. 3. Art. 128. C.E)}$$

Again the sum of (12) & (12') is

$$l + v = \frac{m}{2}\left[\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right] 2\varepsilon^{\frac{h}{2m}}$$

$$l - v = \frac{m}{2}\left[\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right] 2\varepsilon^{-\frac{h}{2m}}.$$

Dividing the first by the last

$$\frac{l+v}{l-v} = \frac{\varepsilon^{\frac{h}{2m}}}{\varepsilon^{-\frac{h}{2m}}} = \varepsilon^{\frac{h}{m}}$$

$$\therefore \frac{h}{m} = \log.\left(\frac{l+v}{l-v}\right)$$

$$\therefore h = m.\log.\left(\frac{l+v}{l-v}\right)$$

whence

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2}\left[m.\log.\left(\frac{l+v}{l-v}\right) + k\right] \\ x_2 &= \tfrac{1}{2}\left[m.\log.\left(\frac{l+v}{l-v}\right) - k\right] \end{aligned}\right\} \cdots\cdots \text{(Eq. 4. Art. 128. C.E)}$$

To compare the catenary with the parabola having a focal distance $= \frac{m}{2}$, expand the eq. of the catenary,

$$y = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right]$$ by McLaurin's Theorem

in which

$$y = (y) + \left(\frac{dy}{dx}\right)\cdot\frac{x}{1} + \left(\frac{d^2y}{dx^2}\right)\cdot\frac{x^2}{1.2} + \left(\frac{d^3y}{dx^3}\right)\cdot\frac{x^3}{1.2.3} + \&c.$$

When $x = 0$

$$(y) = \frac{m}{2}(2) = m$$

$$\left(\frac{dy}{dx}\right) = \frac{m}{2}\left[\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right) = 0\right.$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{m}{2m}\left[\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)\right] = \frac{m}{m^2} = m\cdot\frac{1}{m^2}$$

$$\frac{d^3y}{dx^3} = \frac{m}{2m^2}\left[\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)\right] = 0$$

$$\frac{d^4y}{dx^4} = \frac{m}{2m^3}\left[\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)\right] = m \cdot \frac{1}{m^4}$$

Hence

$$y = m\left[1 + \frac{x^2}{2m^2} + \frac{x^4}{24m^4} + \frac{x^6}{720m^6} + \&c\right]$$

The eq. of a parabola whose focal distance $= \frac{1}{2}$ the modulus of the catenary is

$$x^2 = 2my.$$

with origin at the vertex. Change the origin to the point O at a distance $= m$ below the vertex on the axis y in order to make it coincide with the origin used in the eq. of the catenary, & we have

$$y - m = \frac{x^2}{2m} \quad \text{or}$$

$$y = m\left(1 + \frac{x^2}{2m^2}\right)$$

Differentiating, we obtain

For catenary $-\ \frac{dy}{dx} = \frac{x}{m}\left[1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \&c\right]$

For parabola $-\ \frac{dy}{dx} = \frac{x}{m}.$

To obtain area, multiply value of y by dx & integrate. Then

For catenary $-\ \int y\,dx = mx\left[1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \&c\right]$

For parabola $\int y\,dx = mx\left[1 + \frac{x^2}{6m^2}\right].$

To get lengths –

For catenary $-\ \frac{dy}{dx} = \frac{s}{m} \quad \therefore s = m.\frac{dy}{dx}$

$$\therefore s = x\left[1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \&c\right]$$

The length of a parabolic arc is

$s = \frac{1}{m}\int (m^2 + x^2)^{1/2}dx.$ Expand $(m^2+x^2)^{1/2}$ by the binomial theorem & integrate and

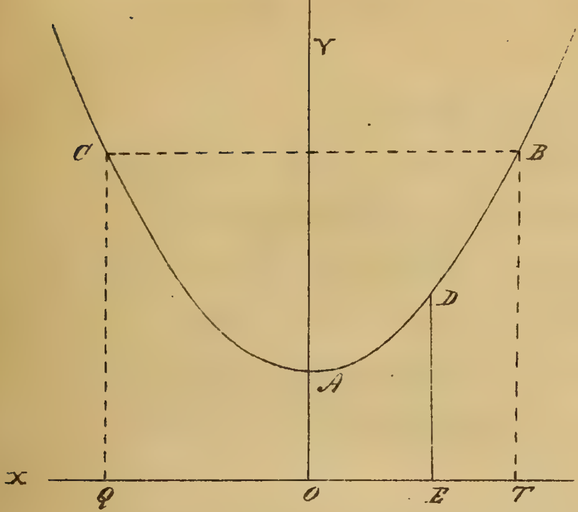
$$s = x\left[1 + \frac{x^2}{6m^2} - \frac{x^4}{40m^4} + \&c\right].$$

— " —

Transformed Catenary.

C.E. Art. 131.

Fig. 1



In the common catenary (Fig. 1) since the area $AOED = ms$ & $\underline{m}$ is constant, the area varies as $\underline{s}$. But the load on the arc AD ($= ps$) also varies as $\underline{s}$, p being constant. Hence a convenient method of representing the load on any arc AD. Suppose a sheet of metal $CQTBAC$ bounded below by the "directrix" QT to be suspended from the curve. Let the wt. of this metal corresponding to $\underline{m}$ units of its surface $= p$; that is, $w\underline{m} = p$ or $w = \frac{p}{m}$. The wt. of a strip a unit in breadth extending from A to O is then $= p =$ wt. of a unit's length of the chain. Then the part of the sheet $AODE$ whose wt. $= wms = ps$ represents ~~the~~ the wt. P on the arc AD. So $AOBT$ represents the wt. on AB & $CQTB$ the whole wt. on CAB.

For the hor. force at A we have

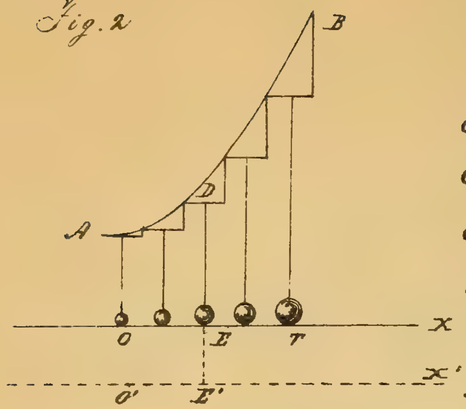
$$H = pm = wm^2 \text{ - - - - - - - - - - - (1)}$$

At any point D, $\quad T = \sqrt{H^2 + P^2} = p\sqrt{s^2 + m^2} = py = wmy$ - - - - (2)

The property above explained may be illustrated in another way.

Construct on AB (Fig. 2.) a series of little triangles with all their bases equal. Let the wt.s of the little arcs constituting the hypothenuses of these triangles be represented by balls suspended by threads from the middle of each arc. Take the length of the thread at A as $= \underline{m}$; make the lengths of all the threads proportional to the wt.s of the balls hung to them. Then the

Fig. 2



lower ends of these threads will all be on the directrix OX (Fig. 2). That is, the intensity of a load on a catenary along the horizontal line (= wt. on a unit of horizontal distance) varies as the ordinates of the catenary when these ordinates are measured from the directrix.

$\therefore$ Intensity $= wy$.

It makes no change in the form of the curve AB to increase or diminish the wt.s provided the proportion among them is preserved. Note, however, that we cannot change the depth AO of the sheet (Fig. 1.) nor the length of the lines (Fig. 2) without changing the curve; for if the lines ended in O'X' for instance instead of OX, then $\frac{AO}{DE}$ would not $= \frac{AO'}{DE'}$.

Hence the modulus ($m = AO$) fixes the catenary; or, if we assume the catenary, the modulus is determined.

This often interferes with the use of the common catenary in the building of arches (in which case the curve is inverted, the metal sheet AOTB is replaced by a wall of uniform material & the tension on the chain CB (Fig. 1) is replaced by a thrust along CAB (Fig. 3).

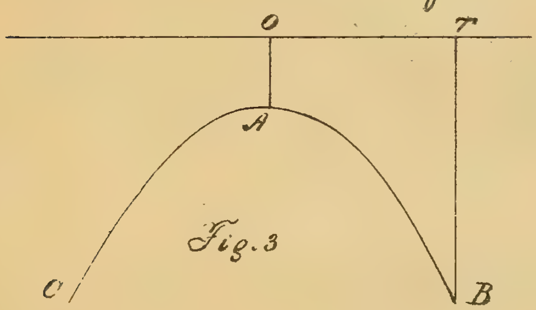


Fig. 3

For we are often compelled to make the curve pass thro. three points while yet the value of AO is fixed.

This difficulty is obviated by the use of the Transformed Catenary which is thus obtained.

Imagine a cylindrical surface constructed upon CATBAC (Fig. 1.) as a base. To simplify matters suppose the elements of the cylinder to be perp. to the plane of the base. Cut this cylinder by a plane inclined to the base, & we shall

get a "transformed catenary" & the shape of the sheet of metal under which it will be balanced; since the new curve & surface cut out by the inclined plane are the parallel projections of the curve CAB & the surface CQIBAC (Fig. 1.). Let this inclined plane be so placed that it shall intersect the base in the straight line CB (Fig. 1.) or in one parallel to it. Then all horizontal lines (those parallel to CB or QI) will be unchanged ~~while~~ in the parallel projection while all vertical lines will be changed in a constant ratio whose magnitude will depend upon the inclination of the cutting plane. Let Fig. 4. be a vertical section of the cylinder on the line OI. Then, if the cutting plane passes through CB, the cylinder reduces to a wedge & the vertical section is the triangle OUI. In this triangle UI is the ordinate of the vertex of the transformed catenary in which the vertical lines are increased in the ratio UI : AO. Laid in the same plane the two curves are CAB & C'A'B' (Fig. 5.).

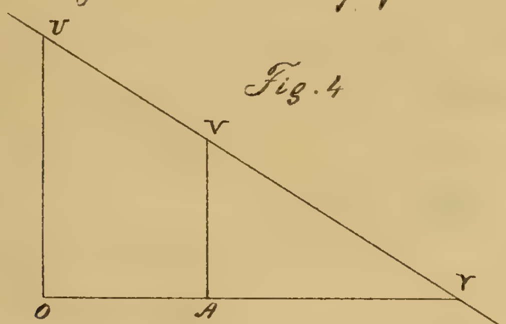
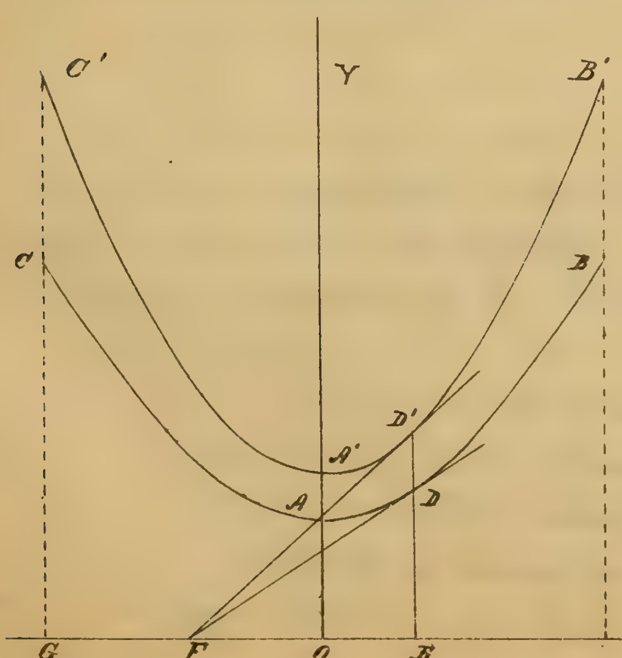


Fig. 4



The eq. of the curve C'A'B' (Fig. 5.) is readily obtained. The abscissas are the same as those in CAB but the ordinates are changed so that (if y' = ordinate of C'A'B' & $y_0 = A'O$)

$$y' : y :: A'O : AO :: y_0 : m$$

$$\therefore y' = \frac{y_0}{m} \cdot y \quad \text{or} \quad y = y' \cdot \frac{m}{y_0}$$

In the eq. of the common catenary, substitute for y the above value & we have

the eq. of $C'A'B'$ $\quad \frac{m}{y_0}\cdot y' = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right]$

$$\therefore\ y' = \frac{y_0}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right] \text{ - - - - - - - - - - - - - - (3)}$$

or $$x' = m.\ hy.\ log.\left[\frac{1}{\frac{y'}{y_0} + \sqrt{\frac{y'^2}{y_0^2} - 1}}\right] \text{ - - - - - - (4)}$$

$$\text{area } A'OED' = \int y'dx = \frac{my_0}{2}\left[\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right] \text{ - - - - - - (5)}$$

&c &c &c

The "triangle of forces" FED for any arc AD of the catenary becomes FED' for the arc $A'D'$ of the transformed cat.^{y} - that is - Since hor. lines & forces are unchanged -

$$\text{Tens.}^n \text{ at vertex } A' = H' = H = wm^2 \text{ - - - - - - - - (6)}$$

The load on $A'D'$ is increased in the ratio $D'E : DE = A'O : AO$.

$$\therefore\ P' = P\frac{D'E}{DE} = P\frac{y_0}{m} \text{ - - - - - - - - - - - - - - (7)}$$

($D'E$ represents this load.)

The tension at $D' = T' = \sqrt{P'^2 + H^2}$ - - - - - - - - - - - - (8)

Intensity of tens.^{n} at $D' = wy'$ - - - - - - - - - - - (9)

$$\text{Tang. of inclination} = \tan i' = \frac{dy'}{dx'} = \frac{y_0}{2m}\left[\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right] \text{ - - - - - (10)}$$

In this curve we can assume the directrix QT (Fig 5), the distance $A'O$ ($= y_0$) & also the points B' & C'. These quantities assumed we determine m (the modulus of the corresponding common cat.^{y}) from eq.(4) & then by Eq.(3) find points of the transformed catenary.

Example. Given span of an arch = 9 ft; rise = 4½ ft; height of brickwork ~~over~~ the crown = 24½ ft ($= y_0$); $w = 112$ lbs per cub. ft. Determine the transformed cat.^{y} & find the amt & direction of the thrust at the abutments.

Ans. - $i = 64°\ 5'$
$H = 6359$ lbs
$P = 13083$ lbs
Radius of curv. at crown = 2.3 ft
" " " " spring.^{g} = 23.46 ft.

Circular Ribs.

C.E. Art. 133.

The uniform normal load on a circular rib is represented in Fig 1, the load on each element $\underline{ds}$ of the curve being constant & perp. to it.

Fig. 1

Note, that the thrust on a rib under any load which is everywhere normal to it must be constant; that is, the compression at A & B & at all other points is the same since none of the elements of the external pressure have tangential components.

Take any two adjoining elements of the rib ds (= DE. Fig. 2) & ds' (= EF. Fig. 2), each of such length as to correspond to an equal element of the load (only when the load is uniform is $ds = ds'$) Represent the little loads on these elements by $p\,ds$ & $p'ds'$. These equal loads being normal to DE & EF respectively, their resultant in the direction OR (Fig. 2) bisects the angle bet. $p\,ds$ & $p'ds'$ & also the angle DEF bet. the directions of the thrusts T & T' on the rib at D & F.

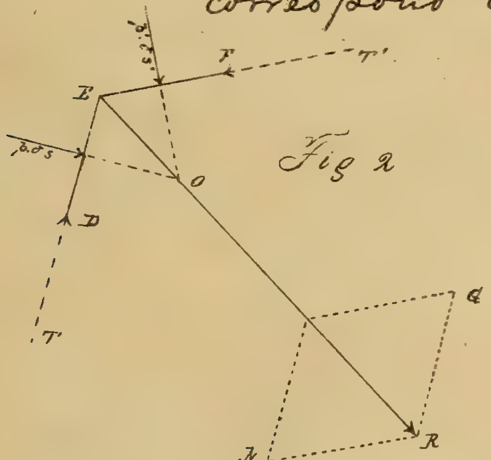


Fig 2

Hence the parallelogram of forces (as shown at R) will be a rhombus, or $NR = T' = RG = T$.

Again, suppose the load uniform & take three elements DE; EF, FH (Fig. 3.) of the rib each bearing the normal load = $p\,ds$. In place of the little arcs use for clearness their chords. Since the loading is uniform, the arcs bearing the equal elements of that load must also be equal or $DE = EF = FH$. We have

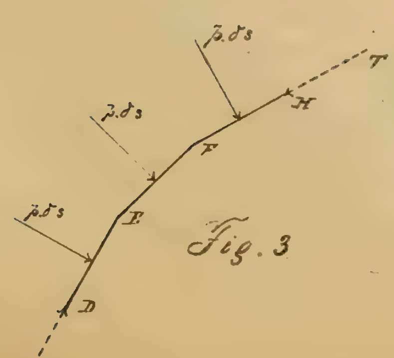


Fig. 3

proved $T = T'$. Hence the sides DE, EF, FH will arrange themselves symmetrically as in Fig. 3. Every other part of the rib containing three elements will assume the same shape exactly as DH, since each such part = DH in length & is acted on by precisely an equal & similar system of forces. Hence the little chords DE &c will constitute a regular polygon: & hence the rib CAB (Fig. 1.) will be the arc of a circle.

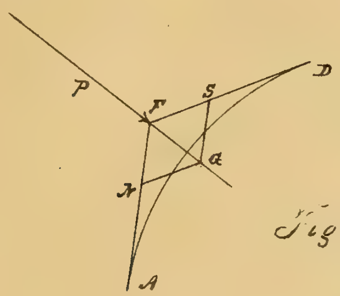


Fig. 4

To form the triangle of forces for any part of a loaded circle as AD (Fig. 4.), draw tangents at A & D & at their intersection lay off FN = FS to represent the equal ~~threads~~ thrusts at A & D. Then FG = the resultant of the load.

It is often easier to deal with a uniform normal load by resolving it into its vert. & hor. components.

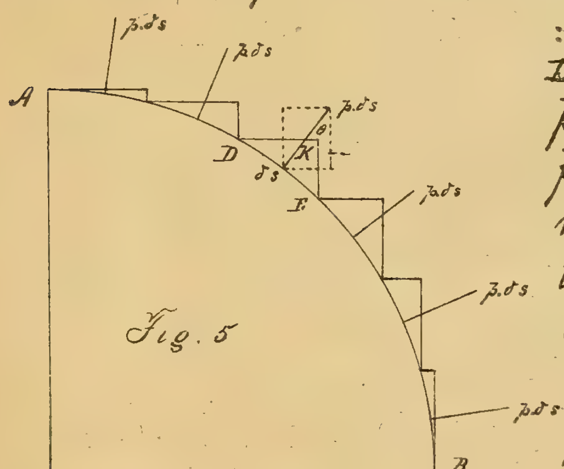
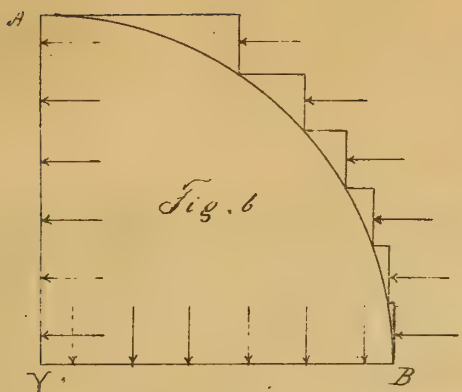


Fig. 5

The load on an element DE (= ds of the quadrant AB) is = $p\,ds$. The hor. comp.^{t} of this = $p.ds.\sin\theta$ where θ = the angle made by the direction of $p\,ds$ with the vert. (= angle made by tang. of ds with horizontal). The vert. component of the little load = $p\,ds.\cos\theta$. Considering the hor. comp.^{t} with reference to the vert. space ($EK = ds.\sin\theta$) over which it is distributed, we have its intensity = $\frac{p\,ds.\sin\theta}{ds.\sin\theta} = p$. So the vert. comp.^{t} is distributed over a hor. space = $DK = ds.\cos\theta$ & $\therefore$ its intensity = $\frac{p\,ds.\cos\theta}{ds.\cos\theta} = p$ = the intensity of the normal force. Hence the normal force at each point is equivalent to a hor. & a vert. force at that point of equal intensity.

If we then construct little triangles on the curve AB (Fig 6) with their vert. sides constant in length



the hor. forces on these sides will be represented by lines of constant length. Transfer these forces to AY which = the sum of all the vert. sides of the little triangles &, as the hor. intensity = p, we have $p(AY)$ = total hor. force on AB. Similarly, if we draw a set of triangles on AB with their hor. sides constant, we see that the total vert. force on $AB = p(YB) = pr$. Hence

1° The resultant of the entire normal force on the quadrant AB = the resultant of a hor. & a vert. force, each of which is $= pr$.

2° The thrust along the rib at A, being = the hor. force on the quadrant AB, is $= pr$. And if T = thrust at any point

$$H = T = pr.$$

—"—

Elliptic Ribs.

C. E. Art. 134.

Fig. 1

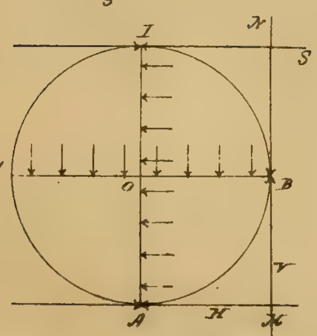
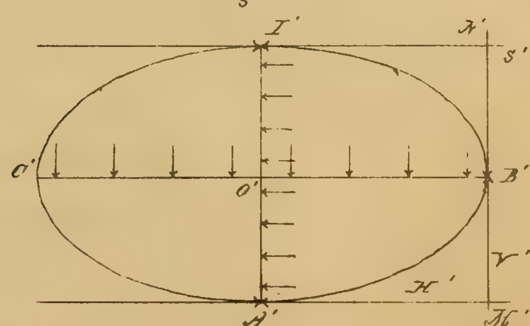


Fig. 2



Erect a cylinder upon the circle (Fig. 1.) as a base & cut it by an inclined plane whose line of intersection

with the plane of the base shall be parallel to AI. The ellipse & forces thus gotten are given in Fig. 2. & are the parallel projections of those in Fig. 1. $A'I' = AI$ while $C'B' = c(CB)$ (where c = the ratio in which the hor. lines have been increased). All vert. lines are unchanged & all hor. ones are changed in the ratio $1:c$.

The circle is the curve assumed by the rib under a uniform hor. & vert. force at each point of the same kind & equal in intensity – for such a system of forces is equivalent to a constant normal force around the curve. For convenience these forces are represented in Fig. 1. along the two diam^s; each little line representing the force on a unit of distance. The thrust around the rib ($= \bar{p}r$) is represented at A & B by the arrows.

In the ellipse, the vert. lines being unchanged, the total vert. force on the elliptic ring is the same as it was in the circle; & if we call the vert. force on a quadrant of the circle P & on a quadrant of the ellipse P', then

$$P = P' \text{ - - - - - - - - - } (1)$$

Note, however, that in the ellipse the force P' is distributed over the distance $O'B'$ & not over a distance OB. Hence the intensity of the force P' or the amt. of it on each unit of distance is not the same. To obtain the intensity of P' we divide it by the space over which it is distributed. Let

$$p_y = \frac{P}{OB} \text{ and } p_x = \frac{H}{AO}.$$

represent the vert. & hor. intensities in the circle where $p_y = p_x = p$. Let

p'_y & p'_x represent the vert. & hor. intensities in the ellipse. Then

$$p'_y = \frac{P'}{O'B'} = \frac{P}{c.OB} = \frac{p_y}{c} \text{ - - - - - - - } (2)$$

The lines representing the thrusts at B & C are also unchanged, & hence the thrusts at B' & C' are $= P' = P$.

The hor. lines are all increased in length in the ratio 1:c. Hence the sum of the lines representing the hor. force on a quadrant of the ellipse is greater than the corresponding sum for the circle in the above ratio

$$\therefore H' = cH \quad \text{-------} \quad (3)$$

The distance over which H' is distributed does not change however, & hence for the intensity of the hor. force in the ellipse we have

$$p_x' = \frac{H'}{A'O'} = \frac{c.H}{AO} = cp_x \quad \text{-------} \quad (4)$$

The hor. thrust in the rib at A' & I' being equal, the hor. force on a quadrant is

$$H' = c.H = c.P = c.P' \quad \text{-------} \quad (5)$$

The thrust around the ellipse is not constant as in the circle. At B' & A' they are as

$$P' : H' :: 1 : c$$

But $A'I' : C'B' :: 1:c$ $\therefore$

1° "The thrusts in an elliptic rib are as the axes to which they are parallel".

Again the intensities in the ellipse are

$$p_y' : p_x' :: \frac{p_y}{c} : cp_x :: \frac{1}{c} : c :: 1 : c^2$$

and $(A'I')^2 : (C'B')^2 :: 1 : c^2$ Hence

2° "The intensities of the forces in the ellipse are as the squares of the axes to which they are parallel."

From the above proportion

$$c = \sqrt{\frac{p_x'}{p_y'}} \quad \text{-------} \quad (6)$$

In the ellipse the resultant of the little hor. & vert. loads at any point is not normal to the curve except at the extremities of the axes.

To determine the relations bet. the forces at other points than the extremities of the axes.

In the circle (Fig. 3.) if we resolve the forces along any two rect. axes as A, I, & C, B, we shall have the same relations as when the forces are resolved along a

Fig. 3

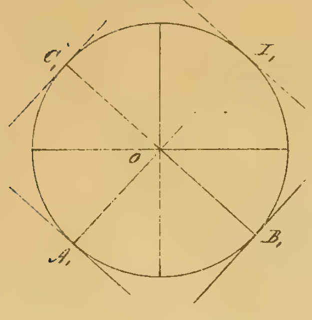
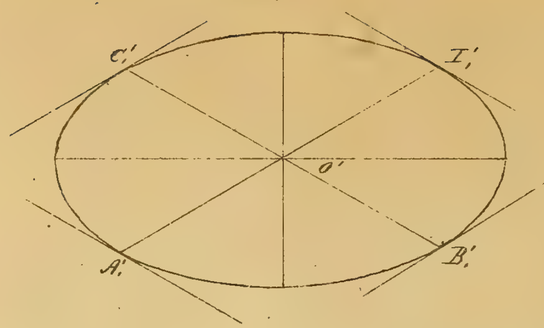


Fig. 4



vert. & hor: axis. Now the three parallel lines viz. the diam. $A_1 I_1$ & the tang^s at C_1 & B_1 are projected in the ellipse (Fig. 4.) into three parallel lines $A_1' I_1'$ & the tang^s at C_1' & B_1'. Similarly $C_1 B_1$ & the tang^s at A_1 & I_1 continue parallel in the ellipse; or rect. diam^s of the circle become conjugate in the ellipse. The lines representing the forces parallel to $A_1 I_1$ in the circle are changed in the ellipse just as $O'I_1'$ is changed from OI_1. Similarly with those parallel to $C_1 O$.

Let $O'I_1' = r'$ and $O'C_1' = r''$ and let the total force parallel to $O'I_1'$ on a quadrant (such as $C_1'I_1'$ or $I_1'B_1'$) of the ellipse be equal P_1 & that parallel to OB_1' be $= H_1$; then, if r = radius of the circle (since the force on any quadrant of the circle $= H = P = T$)

$$\left.\begin{array}{ll} H : H_1 :: r : r'' & \therefore H_1 = \dfrac{Hr''}{r} \\ P : P_1 :: r : r' & \therefore P_1 = \dfrac{Pr'}{r} = \dfrac{Hr'}{r} \\ \therefore H_1 : P_1 :: r'' : r' & \end{array}\right\} \cdots (7)$$

Hence Prop. 1. may be applied generally to all conj. diam^s of the ellipse, that is;

3° The total thrusts along the rib at the extremities of any two conj. diam^s are as the diameters to which they are parallel.

Again the intensities being equal to the total loads divided by the spaces over which they are distributed, let p_{y_1}' = intensity of load parallel to $O'I_1'$ & p_{x_1}' = intensity of that parallel to $O'C_1'$; Then

$$\left.\begin{array}{l} p_{y_1}' = \dfrac{P_1}{O'C_1'} = \dfrac{Pr'}{rr''} = p_y \cdot \dfrac{r'}{r''} \\ p_{x_1}' = \dfrac{H_1}{O'I_1'} = \dfrac{Hr''}{rr'} = p_x \cdot \dfrac{r''}{r'} \\ p_{y_1}' : p_{x_1}' :: p_y \cdot \dfrac{r'}{r''} : p_x \cdot \dfrac{r''}{r'} :: r'^2 : r''^2 \end{array}\right\} \cdots\cdots (8)$$

Hence for Prop. 2. read

4° The intensities of a pair of conj. loads are to each other as the squares of the conj. diam.^s to which they are respectively parallel.

To pass from one set of conj. forces in the ellipse to another; let
p_{y_1} and p_{y} be the intensities & H_1 and P_1 the thrusts parallel to one set of conj. diam.^s & r'' and r' the conj. semi-diam.^s & let p'_{y_1}, p'_{1y_1}, H'_1, P'_1, r''_1, r'_1 be the corresponding quantities of the other set. Then

$$\left.\begin{aligned} & p'_{y_1} = p_y \cdot \frac{r''}{r'} \quad \therefore\ p_y = p'_{y_1} \cdot \frac{r'}{r''} \\ & p'_{1y_1} = p_y \cdot \frac{r''_1}{r'_1} \quad \therefore\ p'_{1y_1} = p'_{y_1} \cdot \frac{r' r''_1}{r'' r'_1} \\ \text{Also} \quad & H_1 = \frac{H r''}{r} \text{ or } H = H_1 \cdot \frac{r}{r''} \\ & \therefore\ H'_1 = \frac{H r''_1}{r} = \frac{H_1 r''_1}{r''} \\ \text{Similarly} \quad & p'_{1y_1} = p'_{y_1} \cdot \frac{r'' r'_1}{r' r''_1} \text{ and } P'_1 = P_1 \cdot \frac{r'_1}{r'} \end{aligned}\right\} \text{---------(9).}$$

[These eq.^s are of use in the distorted elliptic rib (Fig. 1. below) in finding the thrusts at the extremities of the hor. diam.]

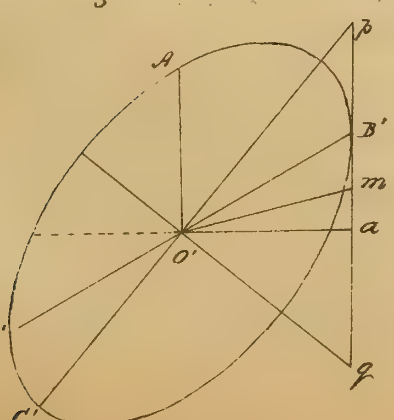
[The ellipse is the form assumed by a rib under a load composed of hor. & vert. comp.^ts which are constant along the hor. & vert. lines but which differ from each other in intensity.]

— " —

C. E. Art. 135.

To find the eq.^s (5) & (6).

Fig. 1



From analyt. geom.^y we have

$$A'B' \sin(\alpha' - \alpha) = AB$$
$$A'^2 + B'^2 = A^2 + B^2$$

In Fig. 1.

$$A'O'p = \alpha \text{ and } C'O'p = \alpha'$$

and $C'O'A' = (\alpha' - \alpha)$ the sine of which is the same as the sine of $A'O'B' = \sin(90^\circ - j) = \cos j$. Hence

$$A'B' \cos j = AB$$

Doubling this eq. & adding it to & subtracting it from the second one above, we get

$$\sqrt{A'^2 + B'^2 + 2A'B'\cos j} = A + B$$

$$\sqrt{A'^2 + B'^2 - 2A'B'\cos j} = A - B$$

$$\therefore \quad A = \tfrac{1}{2}\left[\sqrt{A'^2 + B'^2 + 2A'B'\cos j} + \sqrt{A'^2 + B'^2 - 2A'B'\cos j}\right]$$

$$B = \tfrac{1}{2}\left[\sqrt{A'^2 + B'^2 + 2A'B'\cos j} - \sqrt{A'^2 + B'^2 - 2A'B'\cos j}\right]$$

But $A' = r$ & $B' = cr$

$$\therefore \quad A = \tfrac{r}{2}\left[\sqrt{1 + c^2 + 2c.\cos j} + \sqrt{1 + c^2 - 2c.\cos j}\right]$$

$$B = \tfrac{r}{2}\left[\sqrt{1 + c^2 + 2c.\cos j} - \sqrt{1 + c^2 - 2c.\cos j}\right]$$

The figure constructed as directed in the last paragraph of page 207 gives

$$ap = A \text{ and } aq = B.$$

For, in the first place, $O'p$ and $O'q$ are perp to each other, since the angle $O'mq = 2mO'p$, and, since it is also $= 180^\circ - 2mO'q$, we have

$$2mO'p = 180^\circ - 2mO'q \quad \therefore mO'p + mO'q = 90^\circ$$

Again

$$B'a = \sqrt{(O'a)^2 + (B'O')^2 - 2(B'O')(O'a)\cos j} = \sqrt{r^2 + c^2r^2 - 2cr^2\cos j}$$

Also $pq = 2O'm$, & in the triangle $O'B'a$ we have

$$2(O'm)^2 + 2(B'm)^2 = (O'B')^2 + (O'a)^2 \quad \text{or}$$

$$2(O'm)^2 + \tfrac{1}{2}(B'a)^2 = (O'B')^2 + (O'a)^2$$

$$\therefore \quad (O'm)^2 = \frac{(O'B')^2}{2} + \frac{(O'a)^2}{2} - \frac{(B'a)^2}{4} = \frac{c^2r^2}{2} + \frac{r^2}{2} - \frac{r^2 + c^2r^2 - 2cr^2\cos j}{4}$$

Hence

$$2(O'm) = \sqrt{r^2 + c^2r^2 + 2cr^2\cos j} = pq$$

Now

$$ap = \frac{pq + B'a}{2}$$

$$\therefore \quad ap = \tfrac{1}{2}\left[\sqrt{r^2 + c^2r^2 + 2cr^2\cos j} + \sqrt{r^2 + c^2r^2 - 2cr^2\cos j}\right] = A$$

Similary $\quad aq = \frac{pq - B'a}{2} = rc$

—"—

To obtain eq. 6.

$$\sin B'O'p : \sin O'pB' :: B : cr$$

$$\therefore \sin B'O'p = \frac{B}{cr} \cdot \sin O'pB' = \frac{B}{cr} \cdot \frac{O'q}{pq} = \frac{B}{cr} \cdot \frac{O'q}{A+B} \text{ -----(1)}$$

In the triangle $B'O'q$, we have

$$(B'q)^2 = A^2 = (O'q)^2 + (cr)^2 - 2(cr)(O'q) \sin B'O'p$$

Substituting from (1)

$$A^2 = (O'q)^2 + (cr)^2 - 2B \cdot \frac{(O'q)^2}{A+B}$$

$$\therefore \quad O'q = \sqrt{\frac{(A^2 - c^2r^2)(A+B)}{A-B}}.$$ Substituting this in (1)

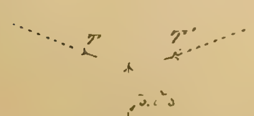
$$\sin B'O'p = \frac{B}{cr} \sqrt{\frac{A^2 - c^2r^2}{A^2 - B^2}}$$

The other expression is found similarly.

—"—

Hydrostatic Arch.
C.E. Art. 136.

In the ellipse (Art. 134.) the action of the load at the extremities of the axes is entirely normal, for at A' & I' the hor. comp.^{t} of the load vanishes & leaves only the vert. comp.^{t} which at those points is normal to the curve. So at C' & B' only the hor. load has a real value & its action is there normal to the curve. Consider the elementary arc ds at A' for instance subjected to this normal load. It is balanced under the equal thrusts $T = T'$ coming from the adjoining parts of the rib & the normal load pds which gives it its curv. Imagine a circle under a constant normal force of intensity $= p$. Take an equal little arc ds with load $\therefore = pds$. Then, if it be acted on at its two ends by thrusts $= T = T'$, it is evident that it will have the same curvature as the arc of the ellipse; or, if it has the same curvature, the thrust around the circle must $= T = T'$



Hence having given the normal load on the curve at any point, we easily determine the thrust along the rib at that point. In the circle

$$H = P = T = p_x r = p_y r = pr.$$

In the ellipse at A' (Fig. 2. Art. 134)

$$H' = p'_y \rho \qquad (\text{where } \rho = \text{rad. of curv.})$$

If $A'O' = r$ & $O'B' = cr$ in the ellipse, we have at A'

$$\rho = \frac{c^2 r^2}{r} = c^2 r$$

$$\therefore \quad H' = p'_y c^2 r = \frac{p_y}{c} \cdot c^2 r = cpr = cH \text{ - - - - - - (1)}$$

So in the parabola under uniform vert. loads,

$$H = 2pm \qquad \text{But}$$

$$H = p\rho = 2pm \qquad (\text{since } \rho = 2m \text{ at the vertex})$$

If the load be everywhere normal to the rib, the above equation will apply to every point, or

$$T = p\rho$$

be a general eq. of the curve; — & further, when the load is everywhere normal, the thrust along the rib must be constant; as there is no tangential force to change it.

$$\therefore \qquad T = p\rho = \text{a constant} \text{ - - - - - - - - (2)}$$

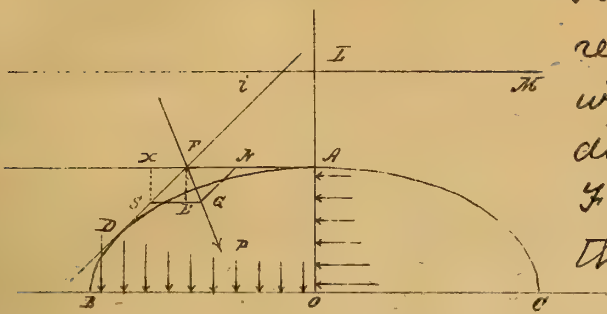
When the load is constant, of course, ρ must be constant too, as in the circle already discussed. When p varies, ρ varies inversely as p which gives curves of which the "hydrostatic arch" is one.

(See text for the equation)

Resolve the normal load on CAB (Fig. 2) as we did in the circle into hor. & vert. comp.s. As in the circle these will be for each point equal in intensity to each other & also to the normal force, or $p = p_x = p_y$. But these quantities are no longer constant all along the curve but vary from point to point.

If we form the parallelogram of forces for

Fig 2



any arc AD (Fig. 2.) with sides $NF = FS$ since $H = T$ = a constant, & FG represents the resultant of the whole load on AD both in amt. & direction. The vert. compt FE of FG = the vert. compt SX of SF; or the vert. load on AD =

$$T \sin i = H \sin i.$$

At B the vert. load $= T = H = Z$ (since $i = 90°$ at that point). So, the hor. compt of the total load on AD is GE; &, since

$$NF = GS = GE + FX \quad , \quad \text{we have}$$

$$\text{Hor. load on } AD = GE = NF - FX$$

$$= H - H \cos i = H(1 - \cos i)$$

At B where $i = 90°$, hor. load on $AB = H$.

On arc DB the hor. load $= H - H(1 - \cos i) = H \cos i$.

The vert. load on AD may be thus expressed

$$H \sin i = \int_0^x p_y \, dx = w \int_0^x y \, dx = w y_0 \rho_0 \sin i$$

The hor. load thus =

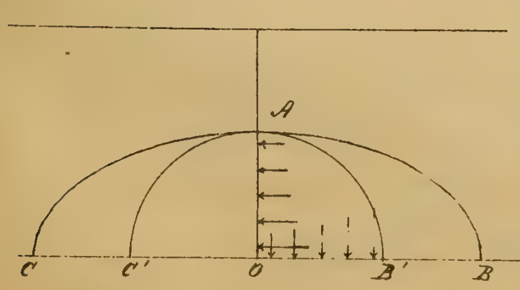
$$H(1 - \cos i) = w y_0 \rho_0 (1 - \cos i) = \int_{y_0}^{y} p_x \, dy = w \int_{y_0}^{y} y \, dy = w \cdot \frac{y^2 - y_0^2}{2}$$

Geostatic Arch.

C.E. Art. 137.

— " —

Fig. 1



If we construct a curve from the last one by using the same ordinates & changing the abscissas in the ratio $c:1$, so that the new coordinates of a point shall be y & cx & at the same time change the horizontal forces in the above proportion leaving the vertical ones unchanged, the new curve and system of forces will be parallel projections of the former ones,

This new curve $C'AB'$ (Fig. 1) is the "geostatic" & bears a relation to the "hydrostatic" analogous to that bet. the ellipse & circle. Hence

$$\left.\begin{array}{l}\text{Total vert. load on } AB' = P' = P = \text{thrust along rib at } B' \\ \text{Total hor. load on } AB' = H' = c.H = \;\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\; A\end{array}\right\} \text{----(1)}$$

The intensities are

$$\left.\begin{array}{l}\text{For vert. load, } p_y' = \dfrac{P'}{OB'} = \dfrac{P}{c.OB} = \dfrac{p_y}{c} \\ \text{" hor. " , } p_x' = \dfrac{H'}{OA} = \dfrac{c.H}{OA} = c p_x\end{array}\right\} \text{------(2)}$$

(P H p_x and p_y referring to the hydrostatic curve)

The load at A & B' & C' being altogether normal (it is not so at the other points), let ρ_0' & ρ_1' be the radii of curv. at A & B'; then

$$H' = p_y' \rho_0' = \frac{p_y}{c}\cdot\rho_0'$$

In the hydrostatic

$$H = p_y \rho_0 \quad \therefore \quad c.H = H' = c p_y \rho_0$$

$$\therefore \quad \frac{p_y}{c}\cdot\rho_0' = c p_y \rho_0 \quad \therefore \quad \rho_0' = c^2 \rho_0 \text{ ---------(3)}$$

So

$$P' = p_x' \rho_1' = c p_x \rho_1' = P$$

In the hydrostatic $P = p_x \rho_1$

$$\therefore \quad p_x \rho_1 = c p_x \rho_1' \quad \therefore \quad \rho_1' = \frac{\rho_1}{c} \text{ ------(4)}$$

These radii are useful in drawing the geostatic curve.

— " —

Linear Rib of any figure.

C.E. Art. 138.

If we assume the curve to be a circle & the vert. load to be uniform in intensity, we know that the hor. load should be also uniform & of an intensity equal to that of the vert. load to produce equilibrium.

But generally.—Let CAB (Fig. 1) be some assumed curve & let the vert. load be known in amt. & distribution. Let

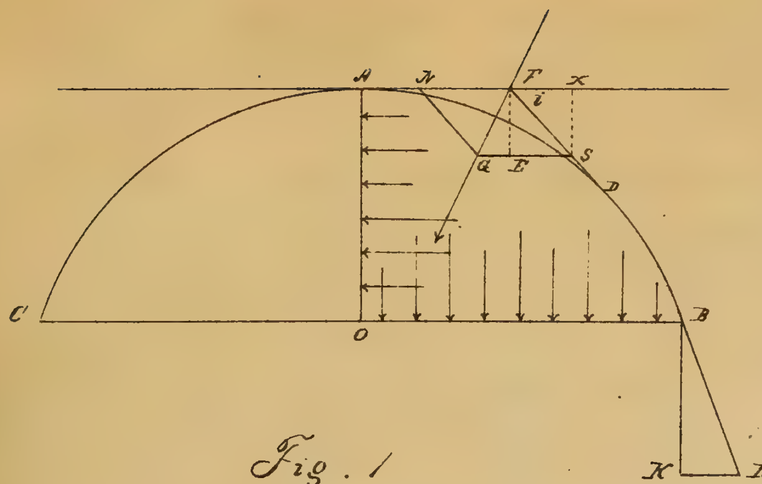


Fig. 1

P = vert. load on any arc AD.
P_1 = " " " semi-rib AB
H = hor. comp.ᵗ of thrust along rib at D = pressure that must be exerted against DB the part of the rib below D.
H_0 = max. value of H
K = hor. load on arc AD
K_1 = " " " semi-rib AB
K_0 = max. value of K.
T = thrust at any point of rib.
T_0 = thrust at A

p_x and p_y = the hor. & vert. intensities.- p_0 = value of p_y at A
ρ_0 and ρ_1 = radii of curv. at A & B.

The vert. load on an arc AD is

$$P = \int_0^x p_y\,dx \text{ ----------- (1)}$$

Again, at the hor. point A the load is entirely vert. &, consequently, normal to the curve. Hence thrust along rib at A is

$$T_0 = p_0\rho_0 \text{ ------------ (2)}$$

To discuss the forces upon the arc AD.— Draw tang.ˢ at A & D: they meet at F (Fig. 1.) thro. which point the resultant of the total load on AD must pass. The vert. comp.ᵗ of that load ($= \int_x^0 p_y dx$) is also = the vert. comp.ᵗ of the thrust along the rib at D, for these two forces, being the only vert. ones connected with AD, must needs balance each other. Therefore, lay off $FN = T_0$ – also lay off FE vert. & $= \int_x^0 p_y dx$. Complete the rectangle $FESX$.

The thrust at D =

$$FS = FE\,\text{cosec}\,i = P\,\text{cosec}\,i = T \text{ ----------- (3)}$$

Also $$SE = FX = P\cot i = H \text{ ----------- ------ (4)}$$

But the hor. thrust at A is

$$T_0 = FN = GS\,,$$

$$\therefore\ GE = T_0 - P\cot i = T_0 - H = K \text{ ----------- (5)}$$

(See text for finding H_0)

The intensity of the hor. load may be expressed thus -

$$p_y = \frac{dK}{dy} = -\frac{d(P\cot i)}{dy} = -\frac{d(P\frac{dx}{dy})}{dy} \text{ - - - - - - - - - - - } (6)$$

At B the vert. load $= P_1$. Represent this by BK (Fig. 1.). If the rib be itself vert. at that point, $BK = P_1$ will be the thrust at B. If the rib be inclined as in the fig. draw its tang. at B and

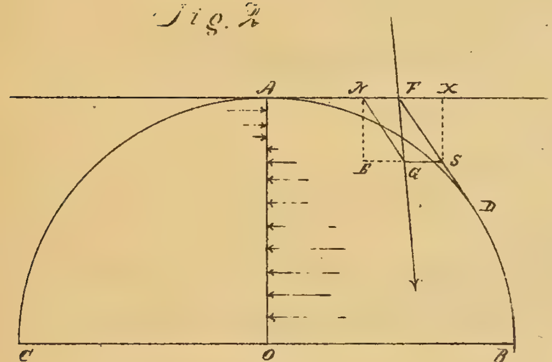
$BL = BK \operatorname{cosec} i_1 = P_1 \operatorname{cosec} i_1$ = thrust along rib.

Then

$KL = BK \cot i_1 = P_1 \cot i_1 = H_1$

$T_0 - P_1 \cot i_1 = K_1$ = entire hor. load on AB.

Fig. 2



It may often happen that $SE = P\cot i$ = hor. compt. of thrust along rib at D (Fig. 2.) is greater than $GS = FN = T_0$ = hor. thrust at A. In such cases, GE $(= T_0 - P_1\cot i_1)$ is negative which indicates that the hor. load bet. A & D for at least a part of the distance must be contrary in character to that heretofore discussed; that is, it must exert an outward instead of an inward thrust (Fig. 2). If this outward thrust were removed or replaced by an inward one, the curve would evidently be flattened about A.

To illustrate geometrically the relations bet. the forces on AB -

The vert. load & curve being given, draw FE^v (Fig. 4.) equal to the total vert. load on AB & lay off on it FE' = vert. load on arc AD'; FE'' = vert. load on AD'' (Fig. 3) &c. Draw a hor. line at F & lay off FN & FK each equal T_0 = thrust at A. Draw thro. F lines parallel to the tang.s at D' D'' D''' &c & thro. E' E'' E''' &c lines parallel to the horizon.

Fig. 3

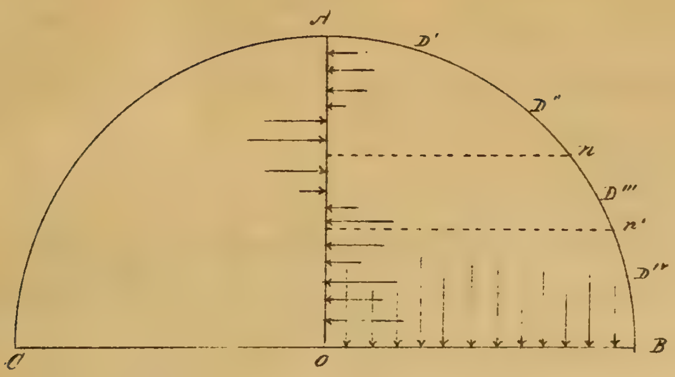
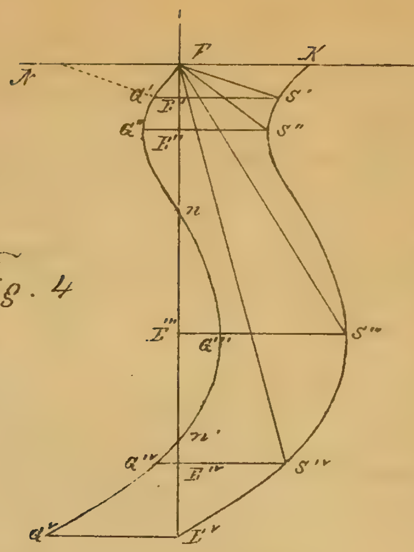


Fig. 4



Then the oblique lines FS' FS'' &c represent the thrusts along the rib at D' D'' &c while E'S' and E''S'' &c represent the horizontal components of those thrusts (= the values of H). Lay off from each point S' S'' &c hor. lines each = FN, & draw thro. the points G' G'' &c thus obtained a curve.

It will evidently be similar to that drawn thro. K S' S'' &c. The line G'E' will represent the resultant of the hor. load that must be distributed along the rib bet. A & D'; G''E'' = horizontal load bet. A & D'' &c. Fig. 4 is really formed from the parallelograms of forces for the arcs AD' &c; this parallelogram being at D' = FNG'S' in which E'S' is the hor. compt. of the thrust at D' & E'G' = resultant of hor. load on AD'. As the abscissas of the curve FG'G'' &c increase to the left of FE^{v} from the point F to G'' (which corresponds to D'' on the curve (Fig. 3.)) the hor. load acts inwardly on the arc AD''. The abscissas then diminish to G'''. Hence bet. D'' & D''' on the curve the hor. load must act outwardly as shown in Fig. 3. From G''' the abscissas increase to G^{v}. Therefore the hor. load acts inwardly on the remainder of the rib. The points n and n' correspond to those arcs on which the resultant of the hor. load = 0. Thus, on the arc An the negative hor. load just equals the positive, & hence their sum = 0. So on the arc An'. Note, that the abscissas of the curve FG'..... G^{v}

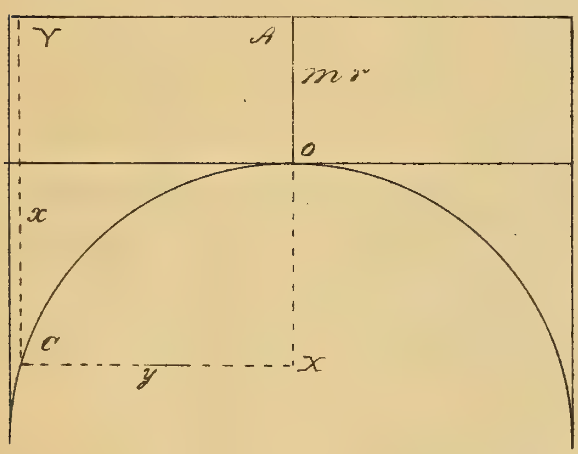
are not the intensities of the hor. loading, but that each such abscissa represents the algebraic sum of the entire hor. load bet. A and the point to which the abscissa corresponds. The intensity in question has been shown already to be $p_x = \frac{dH}{dy} = -\frac{dH}{dy}$.

In this expression, dH = the difference of two neighboring abscissas of the curve FC'----C^v – thus – $dH = C'E' - C''E''$. And dy = vert. projection of the arc $D'D''$ of the rib to which the above correspond.

Eq. 5 (page 216) is merely the eq. for centre of parallel forces.

Ex. 4. Art. 138. C. E.

Let the coordinates be expressed in terms of the inclination of the arch. Then



$x = r(1-\cos i)$; $y = r\sin i$

$dx = r\sin i.di$; $dy = r.\cos i.di$

Thrust at crown

$$T_0 = p_0\rho_0 = (wmr)r = wmr^2$$

The vert. load bet. the crown & any point C is

$$P = w\int(mr+x)\,dy$$
$$= wr^2\int(m+1-\cos i)\cos i.di$$
$$= wr^2\left[m\sin i + \sin i - \int\cos^2 i\,di\right]$$
$$= wr^2\left[(m+1)\sin i - \frac{\cos i\sin i}{2} - \frac{i}{2}\right]$$

From eq. 4. C. E. page 214, the intensity of the hor. pressure is

$$p_y = -\frac{d}{dx}\left(P\frac{dy}{dx}\right) = -\frac{d}{dx}(P\cot i)$$

Substituting $p_y = -\frac{1}{r\sin i}\cdot\frac{d(P\cot i)}{di} =$

$$= -\frac{wr}{\sin i}\cdot\frac{d}{di}\left[(1+m)\cos i - \frac{\cos^2 i}{2} - \frac{i.\cos i}{2\sin i}\right]$$

$$= -\frac{wr}{\sin i}\left[-(1+m)\sin i + \cos i\sin i - \frac{(\cos i - i\sin i)2\sin i - 2i\cos^2 i}{4\sin^2 i}\right]$$

$$= -\frac{wr}{\sin i}\left[-(1+m)\sin i + \cos i \sin i - \frac{\cos i \sin i - i}{2\sin^2 i}\right]$$

$$\therefore \qquad p_y = wr\left[(1+m) - \cos i - \frac{i - \cos i \sin i}{2\sin^3 i}\right]$$

The total hor. pressure in the spandril to any point is

$$P_y = T_0 - P\cot i$$

$$= wmr^2 - \left\{wr^2\left[(m+1)\sin i - \frac{\cos i \sin i}{2} - \frac{i}{2}\right]\right\}\cot i$$

$$= wr^2\left[m - (1+m)\cos i + \frac{\cos^2 i}{2} + \frac{i\cos i}{2\sin i}\right].$$

The hor. comp.^t of the thrust of the arch at any point C is

$$H = T\cos i = T_0 - P_y$$

$$= wr^2\left[(1+m)\cos i - \frac{\cos^2 i}{2} - \frac{i\cos i}{2\sin i}\right]$$ and if C be the point of rupture

$$H_0 = wr^2\left[(1+m)\cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0 \cos i_0}{2\sin i_0}\right]$$

To determine p_y at the crown of the arch, make $i = 0$ in the expression for its value. Then

$$p_y = wr(1 + m - 1 - \tfrac{0}{0}).$$

To find the value of the indeterminate expression $\frac{i - \cos i \sin i}{2\sin^3 i} = \frac{0}{0}$, differentiate and

$$\frac{d}{di}\left[\frac{i - \cos i \sin i}{2\sin^3 i}\right] = \frac{1 + \sin^2 i - \cos^2 i}{6\sin^2 i \cos i} = \frac{0}{0}.$$ Differentiate again & last expression equals

$$\frac{2\sin i\cos i + 2\cos i \sin i}{12\sin i\cos^2 i - 6\sin^3 i} = \frac{0}{0}.$$ Again diff.^g

$$\frac{4\cos^2 i - 4\sin^2 i}{12\cos^3 i - 24\cos i\sin^2 i - 18\sin^2 i\cos i} = \frac{1}{3}$$ Hence

$$p_y = wr(m - \tfrac{1}{3}).$$

From eq. 5. page 216. C.E, since $\cot i$, at springing $= 0$

$$u_H = \frac{\int_{u_0}^{u_1} u p_y \, dx}{H_0} = \frac{\int p_y r[1-\cos i] r\sin i . di}{H_0}$$

$$= \frac{r^2}{H_0}\int_{i_0}^{90^\circ} p_y \sin i (1 - \cos i)\, di$$

Crushing of pillars.

C.E. Art. 158.

Gordon's formula is $P = \dfrac{fS}{1 + a.\frac{l^2}{h^2}}$

The following values of the constants a and f have been obtained by taking the average of those given by several of the best authors.

	f	a
For wood - rect. solid pillars	6000	1/350
" " round " "	6000	4/3(1/350)
For cast iron - solid round pillars	80000	1/325
" " " " rect. "	80000	3/4(1/325)
" " " hollow cylindrical of either square or circular section	80000	1/500
For wrot. iron - solid rect. pillars	36000	1/3000
" " " " round "	36000	4/3(1/3000)
" " " hollow thick cylinders	36000	1/5000

But wrot. iron hollow columns usually buckle. Against that the strength of well-shaped & well-made columns is 27000 pounds per sq. in of section.

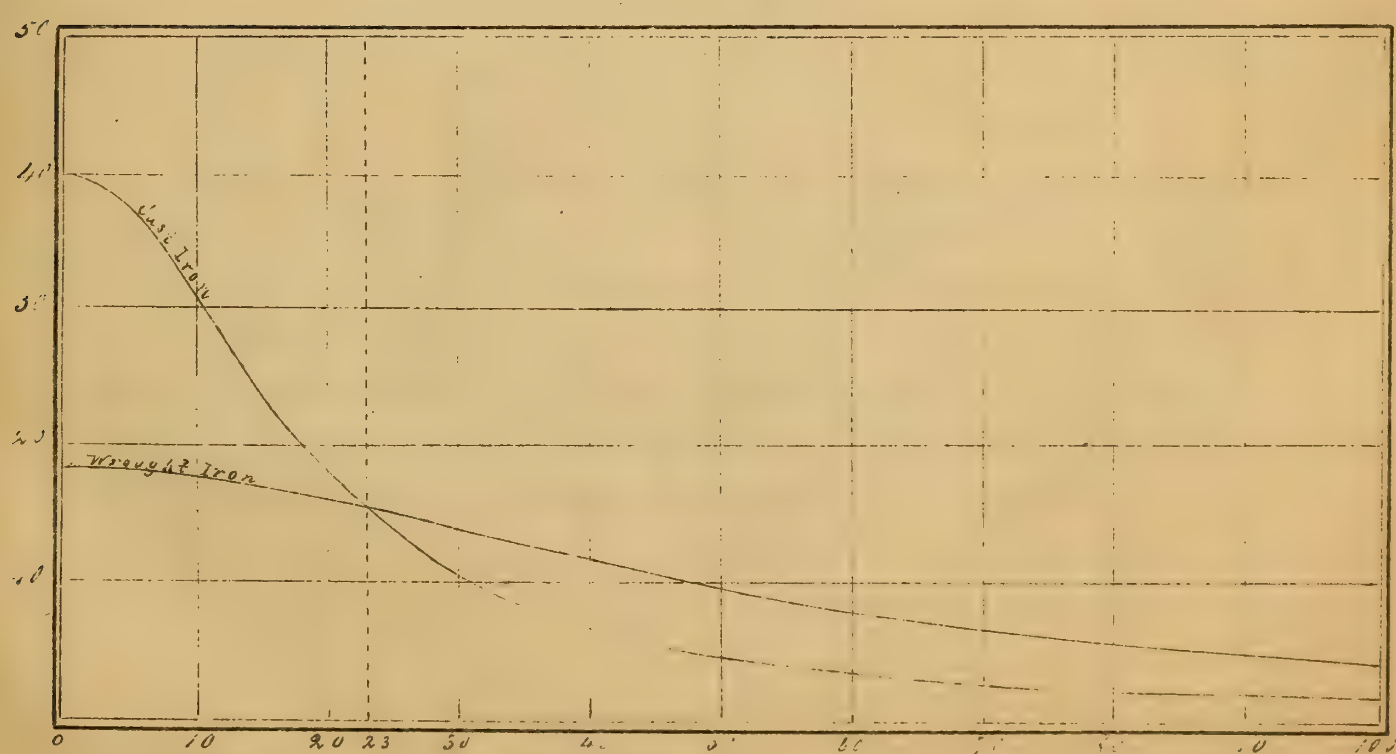
	f	a
For steel (mild) solid round pillars	67000	1/1400
" " " " rect. "	67000	3/4(1/1400)
" " " hollow cylinders	67000	1/2500
For steel (strong) - solid round pillars	114000	1/900
" " " " rect. "	114000	3/4(1/900)
" " " hollow cylinders	114000	1/1500

The above apply to pillars with both ends flat. For pillars with both ends rounded, take 1/3 of the results given by the above formulae. For those with one end rounded, take 2/3.

Writing Gordon's formula as follows—

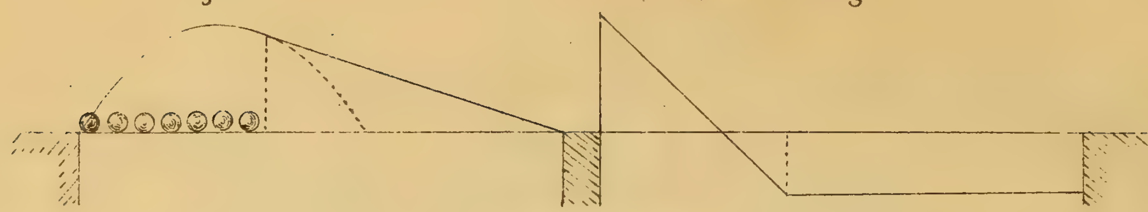
$$\frac{P}{S} = \frac{f}{1 + a_1 \frac{l^2}{h^2}}$$

The right-hand member gives the force per sq. in. of cross-section necessary to crush the pillar; & the relation bet. this force for each material & the value of the ratio $\frac{l}{h}$ may be expressed by a curve. Let the values of $\frac{l}{h}$ be taken as abscissas & the values of $\frac{P}{S}$ as ordinates. The curves below give these relations for a solid cast-iron round pillar & a solid wrot. iron rect. one. If such curves be drawn carefully to a large scale, they may be used in calculating the strength of pillars. Those below show that when $\frac{l}{h} > 23$, the wrot-iron pillar is the stronger, though in short pillars the cast iron has so much the advantage.



C. E. Art. 161. Ex. VII.

Fig. 2 Fig. 1



$$P_1 = \frac{wx''(l - \frac{x''}{2})}{l} \; ; \qquad P_2 = \frac{wx''^2}{2l} \qquad \text{Therefore}$$

$$F = P_1 - \int w\,dx' = \frac{wx''}{l}(l - \frac{x''}{2}) - wx' \quad (\text{bet. } x'' \text{ and origin})$$
$$= -\frac{wx''^2}{2l} \qquad (\text{beyond } x'')$$

$$M = \int F\,dx = \frac{wx''}{l}(l - \frac{x''}{2})x' - \frac{wx'^2}{2} \quad (\text{bet } x'' \text{ and origin})$$
$$= \frac{wx''^2}{2l}(l - x') \qquad (\text{beyond } x'')$$

Fig. 1. shows the shearing force; Fig. 2. the mom^t of the beam. The ordinates in the latter are the values of M.

— " —

C. E. Art. 161. Ex. VIII

The value of F is easily found. To find M.

$$M = \Delta x . \Sigma F.$$

$F = w[\frac{N-1}{2} - n]$ which is the general expression

But $F_0 = w[\frac{N-1}{2}]$; $F_1 = w[\frac{N-1}{2} - 1]$; $F_2 = w[\frac{N-1}{2} - 2]$ &c

Hence the successive values of F constitute a descending arithmetical series whose first term $= w(\frac{N-1}{2})$ & whose last term is $= w[\frac{N-1}{2} - n]$. In finding M for any section, we use n terms of this series & the n^{th} term is $= w[\frac{N-1}{2} - (n-1)]$. Common difference $= -1$

Formula for sum of arith. series is

$$S = \frac{1}{2}n'(2a + (n'-1)d). \quad \text{In this case}$$
$$S = \frac{1}{2}n[N - 1 - n + 1]w = \frac{1}{2}wn[N - n]$$

$\Delta x = \frac{l}{N}$ $\therefore$

$$M = \frac{l}{N} \cdot \frac{1}{2}wn[N - n] = \frac{n(N-n)wl}{2N}.$$

To find $\underline{m}$. When N is even, $n = \frac{N}{2}$

" N " odd, $n = \frac{N-1}{2}$

$\therefore\ m = \frac{M}{Wl}$ in the first case is

$$= \frac{\frac{N}{2}\left(N - \frac{N}{2}\right)wl}{2N} \div w(N-1)l = \frac{\frac{N}{8}}{N-1}$$

& in the latter case,

$$m = \frac{\dfrac{\frac{N-1}{2}\left[N - \frac{N-1}{2}\right]wl}{2N}}{(N-1)wl} = \frac{N^2-1}{8N(N-1)}$$

Rankine obtains his results by making the total load $W = Nw$ instead of $(N-1)w$. By using this value we find in the first case

$$m = \frac{\dfrac{\frac{N}{2}\left[N - \frac{N}{2}\right]wl}{2N}}{Nwl} = \frac{1}{8}$$

In the second case

$$m = \frac{N^2-1}{8N^2}.$$

C. E. Art. 161. Ex. IX.

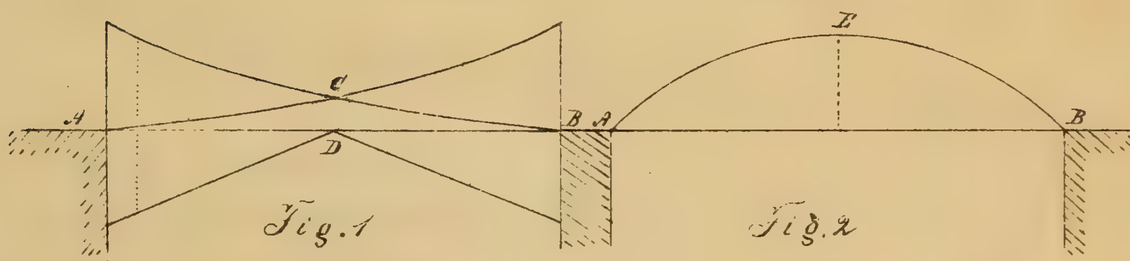
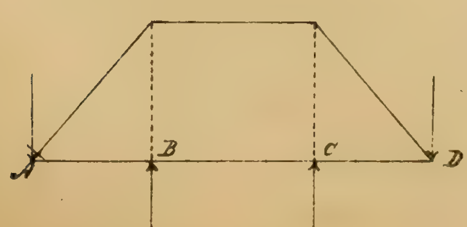


Fig. 1   Fig. 2

Fig. 1. shows the shearing force being the geometrical construction of eq.s (2) & (3). Fig. 2. is a parabola with vertex at E which is the readiest way of constructing eq. 4. In this fig. the ordinates represent the values of M. If the <u>squares</u> of the ordinates represent the values of M, the figure will be the circle given in the text.

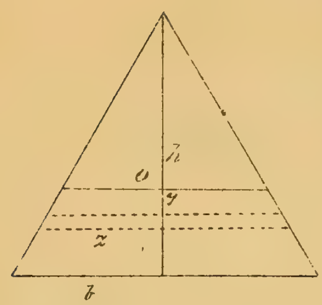
C. E. Art. 161. Ex. X.



The mom.s in this case are shown in the annexed fig. They increase from A to B & D to C and are constant bet. B & C.

C.E. Art. 163. Ex. VII

To find I for an isosceles triangle.



Let O = C. of G. which is at a distance = $\frac{2}{3}h$ from the top & $\frac{1}{3}h$ from the bottom. Then for the breadth at any point

$$b : h :: z : \left(\tfrac{2}{3}h + y\right)$$

$$\therefore\ z = \frac{b\left(\frac{2}{3}h + y\right)}{h}$$

& the area of the little element is

$$= \frac{b\left(\frac{2}{3}h + y\right)}{h} \cdot dy$$

Hence

$$I = \int_{-\frac{2}{3}h}^{\frac{h}{3}} \frac{b\left(\frac{2}{3}h + y\right)}{h} \cdot y^2 dy = \frac{1}{36} bh^3$$

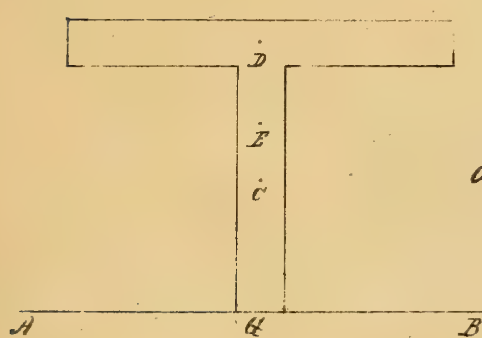
$$\therefore\ n' = \frac{1}{36}$$

and y_1 being the distance from O to the top of the triangle we have

$$m' = \frac{\frac{2}{3}h}{h} = \frac{2}{3} \qquad \therefore\ n = \frac{n'}{m'} = \frac{1}{24}.$$

— " —

C. E. Art. 163. Ex. VIII



In this section take mom.ts around AB. Here $CG = \frac{h_2}{2}$ & $DG = h_2 + \frac{h_1}{2}$

Hence $A_2 \cdot \frac{h_2}{2} + A_1\left(h_2 + \frac{h_1}{2}\right) = Ay$

or $$y = \frac{A_2 h_2 + A_1(2h_2 + h_1)}{2A}$$

$$= \frac{(A_2 + A_1)h_2 + A_1(h_2 + h_1)}{2A} = \frac{Ah_2 + A_1(h_2 + h_1)}{2A} = \frac{(h - h_1)A + A_1(h_2 + h_1)}{2A}$$

$$= \frac{h}{2} + \frac{A_1 h_2 - A_2 h_1}{2A} = \frac{h}{2} + \frac{A_1(h_2 + h_1) - h_1 A}{2A}$$

Again

$$I = \Sigma I' + \Sigma y^2 A'$$

In this case $$\Sigma I' = \frac{A_1 h_1^2}{12} + \frac{A_2 h_2^2}{12}$$

Also

$$\text{wt. } A_1 : \text{wt. } A_2 :: EC : ED \quad \text{whence}$$

$$A_1 + A_2 : A_1 :: EC + ED : EC.$$

$$A_1 + A_2 : A_1 :: \frac{h_1+h_2}{2} : EC = \frac{A_1(h_1+h_2)}{2A}$$

and

$$A_1 + A_2 : A_2 :: EC + ED : ED = \frac{A_2(h_1+h_2)}{2A}$$

hence

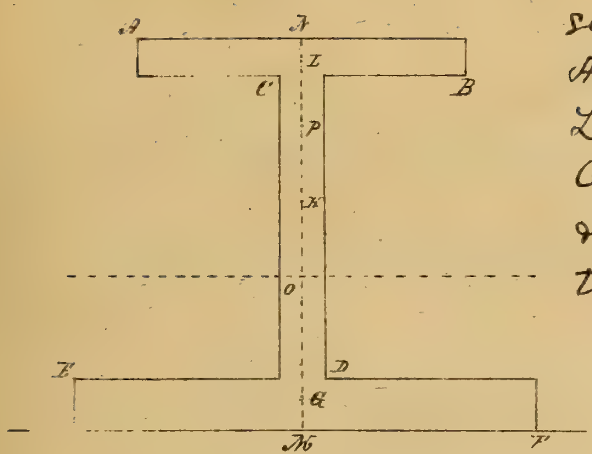
$$\Sigma y^2 A' = A_2\left[\frac{A_1(h_1+h_2)}{2A}\right]^2 + A_1\left[\frac{A_2(h_1+h_2)}{2A}\right]^2$$

$$= \frac{A_2 A_1 (h_1+h_2)^2 (A_1+A_2)}{4A^2} = \frac{A_1 A_2 (h_1+h_2)^2}{4A}$$

$\therefore$ $I =$ &c &c

— " —

C. E. Art. 163. Ex. IX.



Let the fig. represent the section, the upper flange AB being $= A_1$ & $CD = A_2$ & $EF = A_3$. Let L = C. of G. of A_1; H of A_2 & G of A_3. Let P = C. of G. of $(A_1 + A_2)$ & finally O = C. of G. of whole section. Then $LH = \frac{h_1+h_2}{2}$ and

$$A_1 : A_2 :: PH : PL \quad \text{or}$$

$$A_1 + A_2 : A_1 :: \frac{h_1+h_2}{2} : PH$$

$$\therefore PH = \frac{A_1(h_1+h_2)}{2(A_1+A_2)}$$

To find the C. of G. of $(A_1 + A_2)$ and of A_3:
The distance $PG = PH + HG = \frac{h_2+h_3}{2} + \frac{A_1(h_1+h_2)}{2(A_1+A_2)}$

and

$$A_1 + A_2 : A_3 :: OG : PO \quad \text{or}$$

$$A_1 + A_2 + A_3 : A_1 + A_2 :: PG : OG$$

$$\therefore OG = \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A}$$

Hence

$$MO = y_0 = \frac{h_3}{2} + \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A}$$

But $\frac{h_3}{2} = \frac{h}{2} - \frac{h_1+h_2}{2}$ whence

$$y_0 = \frac{h}{2} - \frac{(h_1+h_2)A_3 - (h_2+h_3)A_1 - (h_3-h_1)A_2}{2A}.$$

If find $I = \Sigma I' + \Sigma y^2 A'$.

$$\Sigma I' = \frac{A_1 h_1^2 + A_2 h_2^2 + A_3 h_3^2}{12}$$

For $\Sigma y^2 A'$ we have

$$GH = \frac{h_1+h_2}{2}. \quad HG = \frac{h_2+h_3}{2} \text{ and}$$

$$HG - OG = HO = \frac{h_2+h_3}{2} - \left[\frac{(A_1+A_2)(h_2+h_3)+A_1(h_1+h_2)}{2A}\right]$$

$$GO = GH + HO = \frac{h_1+2h_2+h_3}{2} - [\quad " \quad]$$

$$OG = \frac{(A_1+A_2)(h_2+h_3)+A_1(h_1+h_2)}{2A}$$

HO, GO, OG are the y's needed. Substitute in $\Sigma y^2 A'$ &

$$\Sigma y^2 A' = A_2\left[\frac{(h_2+h_3)^2}{4} - 2\cdot\frac{h_2+h_3}{2}\left[\frac{(A_1+A_2)(h_2+h_3)+A_1(h_1+h_2)}{2A}\right] + (OG)^2\right]$$

$$+ A_1\left[\frac{(h_1+2h_2+h_3)^2}{4} - 2\frac{(h_1+2h_2+h_3)}{2}[\quad " \quad] + (OG)^2\right]$$

$$+ A_3[OG]^2$$ which eq. is equal to

$$(1)\begin{cases} \Sigma y^2 A' = A_2\left\{\dfrac{(h_2+h_3)^2(A_1+A_2+A_3) - 2(h_2+h_3)^2(A_1+A_2) - 2A_1(h_1+h_2)(h_2+h_3)}{4A} + [OG]^2\right\} \\ +A_1\left\{\dfrac{(h_1+2h_2+h_3)^2(A_1+A_2+A_3) - 2(h_1+2h_2+h_3)(h_2+h_3)(A_1+A_2) - 2A_1(h_1+2h_2+h_3)(h_1+h_2)}{4A}\right. + \\ \left. + [OG]^2\right\} + A_3[OG]^2 \end{cases}$$

Now the terms involving $[OG]^2$ when collected are $= A[OG]^2 = \dfrac{[(A_1+A_2)(h_2+h_3)+A_1(h_1+h_2)]^2}{4A}$ ...... (2)

Also the first term in (1) is

$$= \frac{A_2}{4A}\left[(h_1+h_3)^2 A_3 - (h_2+h_3)^2(A_1+A_2) - 2A_1(h_1+h_2)(h_2+h_2)\right]$$

Hence eq. (1) may be written, if we perform the operations indicated in (2), as follows–.

$$\Sigma y^2 A' = \frac{A_2}{4A}\left[(h_2+h_3)^2 A_3 - (h_2+h_3)^2(A_1+A_2) - 2A_1(h_1+h_2)(h_2+h_3)\right]$$

$$+ \frac{A_1}{4A}\left[(h_1+2h_2+h_3)^2(A_1+A_2+A_3) - 2A_1(h_1+2h_2+h_3)(h_1+h_2) - 2(h_1+2h_2+h_3)(h_2+h_3)(A_1+A_2)\right]$$

$$+ \frac{1}{4A}\left[(A_1+A_2)^2(h_2+h_3)^2 + 2A_1(A_1+A_2)(h_2+h_3)(h_1+h_2) + A_1^2(h_1+h_2)^2\right]$$

If we omit the factor $\frac{1}{4A}$ & remember that $(h_1+2h_2+h_3)^2 = [(h_1+h_2)+(h_2+h_3)]^2$, we have the above expression =

$$= A_2 A_3 (h_2+h_3)^2 - A_2^2 (h_2+h_3)^2 - A_1 A_2 (h_2+h_3)^2$$
$$- 2A_1 A_2 (h_1+h_2)(h_2+h_3) + A_1(A_1+A_2)\left[(h_1+h_2)+(h_2+h_3)\right]^2$$
$$+ A_1 A_3 (h_1+2h_2+h_3)^2 - 2A_1(A_1+A_2)\left[(h_2+h_3)^2+(h_1+h_2)(h_2+h_3)\right]$$
$$- 2A_1^2\left[(h_1+h_2)^2+(h_2+h_3)(h_1+h_2)\right] + A_1^2(h_2+h_3)^2 + 2A_1A_2(h_2+h_3)^2$$
$$+ A_2^2(h_2+h_3)^2 + 2A_1(A_1+A_2)(h_2+h_3)(h_1+h_2) + A_1^2(h_1+h_2)^2$$

Collecting ~~&~~ cancelling some of the terms & expanding others the expression becomes

$$A_2 A_3 (h_2+h_3)^2 + A_1 A_3 (h_1+2h_2+h_3)^2$$
$$+ A_1 A_2 (h_2+h_3)^2 - 2A_1A_2(h_1+h_2)(h_2+h_3)$$
$$+ A_1^2(h_1+h_2)^2 + 2A_1^2(h_1+h_2)(h_2+h_3) + A_1^2(h_2+h_3)^2$$
$$+ A_1A_2(h_1+h_2)^2 + 2A_1A_2(h_1+h_2)(h_2+h_3) + A_1A_2(h_2+h_3)^2$$
$$- 2A_1^2(h_2+h_3)^2 - 2A_1A_2(h_2+h_3)^2 - 2A_1^2(h_1+h_2)^2$$
$$- 2A_1^2(h_2+h_3)(h_1+h_2) + A_1^2(h_2+h_3)^2 + A_1^2(h_1+h_2)^2$$
$$= A_2A_3(h_2+h_3)^2 + A_1A_3(h_1+2h_2+h_3)^2 + A_1A_2(h_1+h_2)^2$$

Hence $\quad \Sigma y^2 A' = \frac{1}{4A}\left[\ \&c \quad \&c \quad \&c\ \right].$

C. E. Art. 164. Cases I & II.

Eq. (2)

From eq. (2) page. 255. we have

$$y_a = \frac{h'}{2}\left[\frac{A+A_1}{A}\right]. \quad \text{But } A = A_1 + A_2 \text{ and } h' = y_a + y_b$$

Substituting these values & reducing, we have

$$A_1 = \frac{A_2}{2}\cdot\frac{y_a - y_b}{y_b} = A_2\left(\frac{f_a - f_b}{2f_b}\right) \text{ - - - - - - - - (2)}$$

Also

$$M_0 = \frac{f_a h'}{6}\left[\frac{\left(A_2 + 2A_2\cdot\frac{f_a-f_b}{f_b}\right)A_2}{A_2 + A_2\cdot\frac{f_a-f_b}{f_b}}\right] = \frac{A_2 h'}{6}\cdot(2f_a - f_b) \text{ ..... (3)}$$

Eq. 4. From the first one of Eq.s (4). page. 256. we have

$$y_3 = \frac{h'}{2} \cdot \frac{A_2 + 2A_1}{A} = \frac{y_a + y_3}{2} \cdot \frac{A_2 + 2A_1}{A_1 + A_2 + A_3}$$

Reducing, we get

$$A_3 = \frac{y_a}{y_b} \cdot A_1 + \frac{A_2}{2y_b}(y_a - y_b) = \frac{f_a}{f_b} A_1 + \frac{A_2}{2f_b}(f_a - f_b) \quad \text{------} (4)$$

From the last one of eq. 4. page 256, we have

$$M_0 = \frac{f_b h'}{6} \cdot \frac{A_2(A_2 + 4A_1 + 4A_3) + 12A_1A_3}{A_2 + 2A_1}$$

Substituting value of A_3 from (4) & reducing, we get

$$M_0 = h'\left[\frac{10A_1A_2f_a - A_2^2 f_b - 2A_1A_2f_b + 2A_2^2 f_a + 12A_1^2 f_a}{6(A_2 + 2A_1)}\right]$$

$$= h'\left[\frac{6f_aA_1A_2 + 12A_1^2 f_a + A_2^2(2f_a - f_b) + 2A_1A_2(2f_a - f_b)}{6(A_2 + 2A_1)}\right]$$

$$= h'\left[\frac{f_aA_1(6A_2 + 12A_1) + A_2(A_2 + 2A_1)(2f_a - f_b)}{6(A_2 + 2A_1)}\right]$$

$$= h'\left[f_aA_1 + \frac{A_2}{6}(2f_a - f_b)\right].$$

Substitute value of A_1 from (4) & reduce to get (5)

— " —

C. E. Art. 167.

Let s_1W' = gross safe load of first beam (whose breadth = b').

s_2B' = wt. (multiplied by factor of safety) of first beam. Then $s_1W' - s_2B'$ = net. load and

$$\frac{s_1W'}{s_1W' - s_2B'} = \text{ratio of gross to net load.}$$

Now the wt. of the beam (which is proportional to volume $b'h'l'$), the gross load it can carry ($W = \frac{nfb'h'^2}{ml'}$) & consequently the net load all increase directly as the breadth (b') when the depth (h') & length (l') are unchanged.

We desire to increase the net load from $(s_1W' - s_2B')$ to s_1W' or in the ratio $\frac{s_1W'}{s_1W' - s_2B'}$. Increase the breadth in this ratio & the wt., gross load & net load will all be

increased in the same ratio. Let breadth of new beam = b

Then $b = b' \dfrac{s_1 W'}{s_1 W' - s_2 B'}$. So if W = new gross load

$$W = s_1 W' \frac{s_1 W'}{s_1 W' - s_2 B'} = \frac{s_1^2 W'^2}{s_1 W' - s_2 B'}$$

and wt. of new beam mult.^d by its factor of safety is

$$s_2 B = s_2 B' \cdot \frac{s_1 W'}{s_1 W' - s_2 B'}$$

Then net load of new beam is

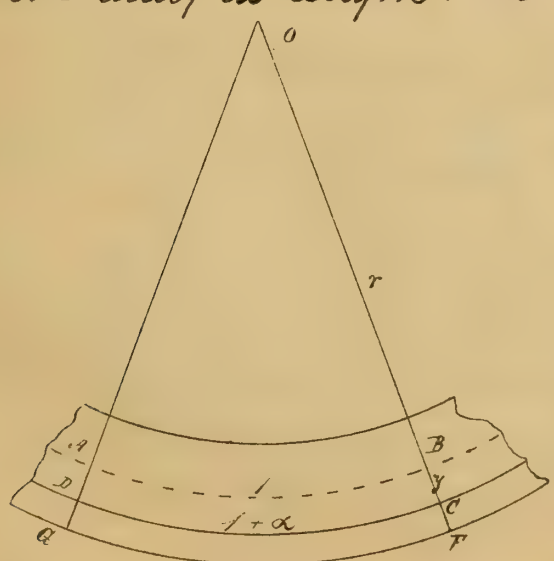
$$W - s_2 B = \frac{s_1^2 W'^2}{s_1 W' - s_2 B'} - \frac{s_2 B' \cdot s_1 W'}{s_1 W' - s_2 B'} = \left(\frac{s_1 W' - s_2 B'}{s_1 W' - s_2 B'}\right) s_1 W' = s_1 W'$$

— " —

C. E. Art. 169

Deflection usually means the greatest displacement of any point of a loaded beam from its position when the beam is unloaded. As the eq.^s relating to deflection involve the laws of elasticity (which laws are only true within elastic or proof limits) these eq.^s apply only to deflections produced by loads not exceeding proof. We divide the process of finding deflection into three steps – I – Curvature ; II – Slope ; III – Deflection.

I – Curvature – Take the little arc AB along the neut. axis = unity in length. Let r = rad. of curv. Let BC = distance from the neutral axis to any other fibre = y & $1+\alpha$ = length of that fibre as extended or compressed.



Then $r : 1 :: r+y : 1+\alpha \therefore r = \dfrac{y}{\alpha}$ or the curv. $= \dfrac{1}{r} = \dfrac{\alpha}{y}$.

Now α = the strain or proportional amt. of lengthening due to the stress on DC, & since the strain is directly proportional to the force producing it, if we call p the force of extension or compression acting on a fibre DC, we will have

$$\alpha : p :: 1 : E \quad \therefore \alpha = \frac{p}{E}.$$

(Since E or the modulus of elasticity is the force which would

stretch DC to double its length & consequently, give a strain $=1$). Substitute this in the value of $\frac{1}{r}$ and we have

$$\frac{1}{r}=\frac{p}{Ey}$$

If we make $y=$ the distance to the outside fibre, it becomes y_1 & p then $=f=$ force on outside fibre & $\therefore$

$$\frac{1}{r}=\frac{f}{Ey_1} \quad \text{-------------- (1)}$$

If we make $f=f'$ the proof stress & call the curv. at the point of greatest strain $\frac{1}{r_0}$, we will have by putting for y_1 its value $m'h$

$$\frac{1}{r_0}=\frac{f'}{Em'h} \quad \text{----------- (2)}$$

Eq.^{s} (1) & (2) give us a value of $\frac{1}{r}$ in terms of the force f & depth h. Let us find a value involving the bending mom.^{t} which is often most convenient for calculation.

From the expression for the mom.^{t} of resistance,

$$M=\frac{fI}{y_1} \quad \text{whence} \quad f=\frac{My_1}{I}. \quad \text{Substitute}$$

in (1) &

$$\frac{1}{r}=\frac{M}{EI} \quad \text{-------------- (3)}$$

If $M_0=$ the bending mom.^{t} at the section of greatest stress & $i_0=$ mom.^{t} of inertia at that section, then

$$\frac{1}{r_0}=\frac{M_0}{EI_0} \quad \text{---------- (4)}$$

From (3) & (4) $\quad \frac{1}{r}:\frac{1}{r_0}::\frac{M}{EI}:\frac{M_0}{EI_0} \quad \therefore \frac{1}{r}=\frac{M}{EI}\cdot\frac{1}{r_0}\cdot\frac{EI_0}{M_0}$

But from (2) $\quad \frac{1}{r_0}=\frac{f'}{Em'h} \quad \therefore \frac{1}{r}=\frac{f'}{Em'h}\cdot\frac{MI_0}{IM_0} \quad \text{---------- (5)}$

or again $\quad \frac{1}{r_0}=\frac{M_0}{EI_0} \quad \therefore \frac{1}{r}=\frac{M_0}{EI_0}\cdot\frac{MI_0}{IM_0} \quad \text{---------- (6)}$

Cor. 1. From the first expression we deduced for the curv, ($\frac{1}{r}=\frac{\alpha}{y}$), it is plain that the curv. is directly as the strain & inversely as y (which last depends on h & the figure of cross section), & when $\frac{1}{r}$ becomes $\frac{1}{r_0}$, $\alpha=$ the proof strain.

Cor. 2. In (3), if the bending moment M be regarded as the same for two beams, then, since E is constant for the same material, we shall have $\frac{1}{r}\propto\frac{1}{I}$ and, if we

consider only beams of similar section, $I \propto bh^3$

$$\therefore \frac{1}{r} \propto \frac{1}{bh^3} \quad \text{or}$$

"In two beams of the same material at sections of similar figures, when the bending mom.ts are equal, the curvatures are inversely as the breadths and the cubes of the depths".

Cor. 3. If instead of making the M's the same for the two beams, we make the loads equal, we have, since $M \propto Wl$ and W becomes constant by this latter supposition,

$$\frac{1}{r} \propto \frac{l}{bh^3}$$

Cor. 4. If in (2) we have f' the same for the two beams & these beams have moreover similar sections so that m' be the same for both, then $\frac{1}{r_0} \propto \frac{1}{h}$ or

"In two beams of the same material at sections of similar figures, when the proof stresses are equal, the curvatures are inversely as the depths."

Cor. 5. In the case of beams of cross-sections of equal strength, eq. (2) may be thus transformed.

In $\frac{1}{r_0} = \frac{f'}{Ey_1}$, the f' is either f_a or f_b (page. 256. C.E) according to circumstances, and

$$\frac{f'}{y_1} = \frac{f_a}{y_a} \text{ or } = \frac{f_b}{y_b}.$$ But from eq. (4) page. 256

$f_a + f_b : f_a :: h : y_a$ $\quad \therefore \frac{f_a}{y_a} = \frac{f_a + f_b}{h}$

Substituting, we have

$$\frac{1}{r_0} = \frac{f_a + f_b}{Eh} \text{ - - - - - - - - - - - - } (7)$$

II. Slope - is the difference of inclination of any two parts of the beam, & is measured by the angle included bet. the tang.s at those points.

Thus for any differential part of the beam as AB the slope is the angle $\underline{di}$ included bet. the tang.s AC & BD; &, if O be the centre of the osculatory circle, then the angle at O contained by the

radii AO & BO also = di and, if AB = dx, we have $r\,di = dx$ whence $di = \frac{dx}{r}$ - - - - - - - - - - - - - - - - - - (8)

Integrate &

$$i = \int \frac{dx}{r} \text{ - - - - - - - - - - - - (9)}$$

This is the expression for the difference of inclination of any two points in a beam. If these inclinations be referred to any fixed standard as for instance a hor. line, let i_0 = the inclination of the beam at one of the points to this line & then

$$i = i_0 + \int_0^x \frac{dx}{r} \text{ - - - - - - - - - (10)}$$

is the general expression for the inclination to the horizon. When at one of the points the beam is hor. let this point be the origin, & then $i_0 = 0$, & the eq. becomes again

$$i = \int_0^x \frac{dx}{r} \text{ - - - - - - - - - - - - - - (11)}$$

Cor. 1. Eq. 11. shows that the slopes are as the lengths & curvatures.

Cor. 2. Under equal bending mom.ts Cor. 1. above & Cor. 2. under (I) give

$$i \propto \frac{l}{bh^3}.$$

Cor. 3. When the beams are of the same material & of similar section the above Cor. 1. in conjunction with Cor. 3. of (I) gives, when the loads ~~are~~ equal,

$$i \propto \frac{l^2}{bh^3}.$$

Cor. 4. Under the same conditions, Cor. 4. of (I) & Cor. 7 above give

$$i \propto \frac{l}{h} \text{ under proof loads.}$$

Cor. 5. To obtain the value of the steepest slope in the ordinary cases. Substitute in the eq. $i = \int \frac{dx}{r}$ the value of $\frac{1}{r}$ from (6) and

$$i = \int \left(\frac{M I_0}{I M_0}\right) \frac{M_0}{E I_0} \cdot dx$$

In this expression $\frac{M_0}{EI_0}$ is a constant & $\frac{MI_0}{IM_0}$ is a numerical ratio (since $\frac{M}{M_0}$ must be such a ratio and $\frac{I_0}{I}$ also) The integral of $\frac{MI_0}{IM_0}dx$ must ∴ = (S)c where S is a numerical factor & c is the integral

of dx; for if, for sake of illustration, we suppose the expression $\int \frac{MI_0}{IM_0}\, dx$ to be composed of the sum of the little quantities $\frac{1}{2}\Delta x + \frac{1}{3}\Delta x + \frac{1}{6}\Delta x + \&c$ where $\frac{1}{2}\ \frac{1}{3}\ \frac{1}{6}$ &c represent the various values of $\frac{MI_0}{IM_0}$, we shall have

$$\Sigma \frac{MI_0}{IM_0}.\Delta x = (S)\,\Sigma.\Delta x = (S)c.$$

Hence the eq.

$$i = \int \frac{MI_0}{IM_0}\frac{M_0}{EI_0}.\, dx = \frac{M_0}{EI_0}\int \frac{MI_0}{IM_0}.\, dx$$

$$= \frac{Wl}{En'bh^3}.\, m\int \frac{MI_0}{IM_0}.\, dx = \frac{Wl}{En'bh^3}.\, mm''c$$

(by placing m'' = the numerical factor above indicated by S)

$$= \frac{m'''Wc^2}{En'bh^3}$$ (by putting $mm'' = m'''$ when $l = c$ & $mm'' = \frac{m'''}{2}$ when $l = 2c$)

$$\therefore\ i_1 = \frac{m'''Wc^2}{En'bh^3} \text{ - - - - - - - (12)}$$

Again using the value of $\frac{1}{r}$ in (5) in eq. (11), we have

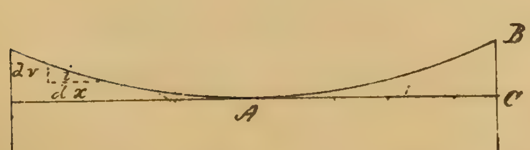
$$i = \int \frac{f'}{Em'h}.\frac{MI_0}{IM_0}.\, dx = \frac{f'}{Em'h}\int \frac{MI_0}{IM_0}.\, dx = \frac{f'}{Em'h}.\, m''c.$$

$$\therefore\ i_1 = \frac{m''f'c}{Em'h} \text{ - - - - - - - - - - (13)}$$

Or, since $\frac{f'}{Em'h} = \frac{f'}{Ey_1} = \frac{f_a}{Ey_a} = \frac{f_a + f_b}{Eh}$, we may write (13) as follows

$$i_1 = \frac{m''(f_a + f_b)c}{Eh} \text{ - - - - - - - - (14)}$$

III - Deflection. = depression of the lowest point below the highest point of an originally straight hor. longitudinal line in the beam, or it is the max. ordinate CB of the curve AB representing the beam in its bent position



Calling this v, we have in the differential triangles

$$\tan i = \frac{dv}{dx}.$$

In practice the curve of deflection has so little curv. that the tang. differs very little from the arc & ∴ we may say $i = \frac{dv}{dx}$ or $dv = i\,dx$ or $v = \int i\,dx$ ---------- (15)

In a beam supported at both ends & symmetrically loaded, since the hor. point (which is the origin) will be at the middle point, we will have

$$v = \int_0^{\frac{1}{2}l} i\,dx$$

and when the beam is fixed at one end & loaded at the other

$$v = \int_0^l i\,dx$$

Cor. 1. Eq. (15) shows that the deflections are as the slopes & lengths.

Cor. 2. This with Cor 3. of (II) gives

$$v \propto \frac{l^3}{bh^3}$$

when the same conditions are imposed on the beam as before.

Cor. 3. With Cor 4 of (II), Cor. 1. above gives

$$v \propto \frac{l^2}{h}$$

Cor. 4. Substitute in (15) & we have

$$v_1 = \iint \frac{MI_0}{IM_0} \cdot \frac{M_0}{EI_0}\,dx^2 = \frac{mWl}{En'bh^3} \iint \frac{MI_0}{IM_0}\,dx^2$$

$$= \frac{Wl}{En'bh^3} \cdot m \iint \frac{MI_0}{2\,IM_0} \cdot 2\,dx^2 = \frac{Wl}{En'bh^3} \cdot mn''c^2$$

$$\therefore \quad v_1 = \frac{Wc}{En'bh^3} \cdot n'''c^2 = \frac{n'''Wc^3}{En'bh^3} \text{ ------- (16)}$$

(by placing $n''' = mn''$ when $l = c$ & $\frac{n'''}{2} = mn''$ when $l = 2c$.) So if we use the other value of i

$$v_1 = \iint \frac{MI_0}{IM_0} \cdot \frac{f'}{Em'h} \cdot dx^2 = \frac{f'}{Em'h} \iint \frac{MI_0}{IM_0}\,dx^2$$

$$= \frac{f'}{Em'h} \cdot n''c^2 \text{ ---------- (17)}$$

Similarly we obtain

$$v_1 = \frac{n''(f_a + f_b)c^2}{Eh} \text{ ------------ (18)}$$

C. E. Art. 169 (Table)

A. Uniform cross-section or $\frac{I}{I_0} = 1$

1° Constant moment of flexure.

$$\frac{M}{M_0} = 1 \quad \therefore m''c = \int_0^c \frac{M}{M_0}\,dx = \int_0^c dx = c$$

$$\therefore m'' = 1 \quad . \quad \text{Also}$$

$$n''c^2 = \int_0^c\int_0^x \frac{M}{M_0}\,dx^2 = \frac{c^2}{2} \quad \therefore n'' = \frac{1}{2}$$

2° Fixed at one end & loaded at the other.

$$M = W(-c+x) \quad . \quad M_0 = -cW \quad \therefore \frac{M}{M_0} = 1 - \frac{x}{c}$$

$$m''c = \int_0^c \left(1 - \frac{x}{c}\right)dx = c - \frac{c}{2} = \frac{c}{2} \quad \therefore m'' = \frac{1}{2}$$

$$n''c^2 = \int_0^c\int_0^x \left(1 - \frac{x}{c}\right)dx^2 = \int_0^c \left(x - \frac{x^2}{2c}\right)dx = \frac{c^2}{2} - \frac{c^2}{6} = \frac{1}{3}c^2$$

$$\therefore n'' = \frac{1}{3} \quad . \quad m''' = mm'' \quad \text{but } m = 1$$

$$\therefore m''' = \frac{1}{2} \quad . \quad n''' = mn'' = \frac{1}{3} .$$

3° Fixed at one end & uniformly loaded.

$$M = -\frac{w(c-x)^2}{2} \quad , \quad M_0 = -\frac{wc^2}{2} \quad \therefore \frac{M}{M_0} = \left(1 - \frac{x}{c}\right)^2$$

$$m''c = \int_0^c \left(1 - \frac{2x}{c} + \frac{x^2}{c^2}\right)dx = c - \frac{c^2}{c} + \frac{c^3}{3c^2} = \frac{1}{3}c \quad \therefore m'' = \frac{1}{3}$$

$$n''c^2 = \int_0^c\int_0^x \left(1 - \frac{2x}{c} + \frac{x^2}{c^2}\right)dx^2 = \frac{c^2}{2} - \frac{c^3}{3c} + \frac{c^4}{12c^2} = \frac{1}{4}c^2$$

$$\therefore n'' = \frac{1}{4}$$

$m''' = mm''$. In this case $m = \frac{1}{2} \therefore m''' = \frac{1}{6}$

So $n''' = mn'' = \frac{1}{8}$

4° Supported at both ends, loaded in the middle.

$$M = \frac{c-x}{2}W . \quad M_0 = \frac{cW}{2} \quad \therefore \frac{M}{M_0} = 1 - \frac{x}{c}$$

$$m''c = \int_0^c \left(1 - \frac{x}{c}\right)dx = \frac{1}{2}c \ . \ \therefore m'' = \frac{1}{2}$$

$m''' = 2mm'' = \frac{1}{4}$. since in this case $m = \frac{1}{4}$

$$n''c^2 = \int_0^c\int_0^x \left(1 - \frac{x}{c}\right)dx^2 = \frac{1}{3}c^2 \quad \therefore n'' = \frac{1}{3}$$

and $n''' = 2mn'' = \frac{1}{6}$

5° Supported at both ends & uniformly loaded.

$$M = \frac{w(c^2 - x^2)}{2}, \quad M_0 = \frac{wc^2}{2} \quad \therefore \frac{M}{M_0} = 1 - \frac{x^2}{c^2}$$

$$m''c = \int_0^c \left(1 - \frac{x^2}{c^2}\right)dx = \frac{2}{3}c \quad \therefore m'' = \frac{2}{3}.$$

$m''' = 2mm''$. But here $m = \frac{1}{8}$ $\therefore m''' = \frac{1}{6}$

$$n''c^2 = \int_0^c\int_0^x \left(1 - \frac{x^2}{c^2}\right)dx^2 = \frac{5}{12}c^2 \quad \therefore n'' = \frac{5}{12}$$

$$n''' = 2mn'' = \frac{5}{48}$$

B Uniform strength & uniform depth. In these $\frac{M}{I}$ is constant (since $M = \frac{f}{y}I$ & $\therefore \frac{M}{I} = \frac{f}{y}$ = a constant for uniform strength & depth) $\therefore \frac{MI_0}{IM_0} = 1.$

6° $m''c = \int_0^c dx = c \quad \therefore m'' = 1$

$$n''c^2 = \int_0^c\int_0^x dx^2 = \frac{c^2}{2} \quad \therefore n'' = \frac{1}{2}.$$

But $m = 1$ $\therefore m''' = 1$ and $n''' = \frac{1}{2}$

7° Fixed at one end & uniformly loaded.

m'' & n'' as in 6°.

Here $m = \frac{1}{2}$ $\therefore m''' = \frac{1}{2}$ & $n''' = \frac{1}{4}$

8° Supported at both ends, loaded in the middle.

m'' & n'' as in 6°. $m = \frac{1}{4}$ $\therefore m''' = 2mm'' = \frac{1}{2}$

$$n''' = 2mn'' = \frac{1}{4}$$

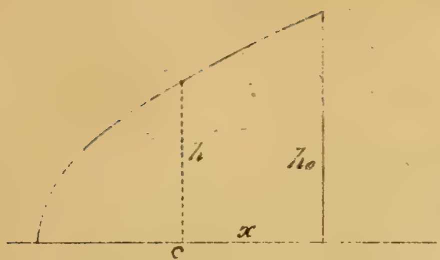
9° Supported at both ends uniformly loaded.

m'' & n'' as in 6°. $m = 1/8$

$m''' = 2mm'' = \frac{1}{4}$. $n''' = 1/8$

C Uniform strength - uniform breadth. This gives $f' : h :: f : y$ where f' = stress at outside fibre whose distance from neutral axis $= h$; & f = stress at any other fibre whose distance $= y$. $\therefore f' = \frac{fh}{y}$. But by supposition above f' is to be constant. Now $\frac{Mh}{I} = \frac{fh}{y}$ = a constant (since $\frac{fh}{y}$ is constant) Hence $\frac{Mh}{I} = \frac{M_0 h_0}{I_0}$ or $\frac{MI_0}{IM_0} = \frac{h_0}{h}$

10° Fixed at one end & loaded at the other.



$h^2 : h_0^2 :: c-x : c \quad \therefore \frac{h_0}{h} = \sqrt{\frac{c}{c-x}}$

$m''c = \int_0^c \sqrt{\frac{c}{c-x}} \cdot dx = 2\sqrt{c}\sqrt{c} = 2c \quad \therefore m'' = 2$

$n''c^2 = \int_0^c\int_0^x \sqrt{\frac{c}{c-x}}\, dx^2 = \sqrt{c}\int_0^c\int_0^x (c-x)^{-\frac{1}{2}} dx^2$

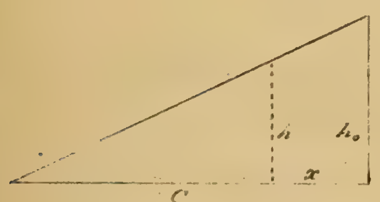
But $\int_0^x (c-x)^{-\frac{1}{2}} dx = -2(c-x)^{1/2}$ when $x = x$.

when $x=0$, $\sqrt{c}\int_0^x (c-x)^{-\frac{1}{2}} dx = 2c^{1/2} \cdot c^{1/2} = 2c$

$\therefore n''c^2 = -\int_0^c 2\sqrt{c}(c-x)^{1/2} dx + \int_0^c 2c\, dx$

$= -\frac{4}{3}c^2 + 2c^2 = \frac{2}{3}c^2 \quad \therefore n'' = \frac{2}{3}. \quad m = 1 \therefore m''' = 2 \;\&\; n''' = \frac{2}{3}$

11° Fixed at one end & uniformly loaded.



$h : h_0 :: (c-x) : c \quad \therefore \frac{h_0}{h} = \frac{c}{c-x}$

$m''c = c\int_0^c \frac{dx}{c-x} = -c \cdot \log.(c-x) + b =$

$-c \cdot \log. 0 + c \log c = -c(-\infty) + c \log. c = \infty$

$m''' = m m'' = \infty$

$n''c^2 = \int_0^c\int_0^x \frac{c}{c-x} \cdot dx^2 = c\int_0^c (\log c - \log.(c-x))\, dx$

$= c^2 \log c - c\int_0^c \log.(c-x)\, dx$. Integrating this last by parts place $y = \log.(c-x)$. $\therefore dy = -\frac{dx}{c-x}$. Place $dz = dx \therefore z = x$

$n''c^2 = c^2\log c - c\left[x \log.(c-x) + \int_0^c \frac{x\,dx}{c-x}\right] = c^2 \log. c - c^2 \log.(c-c) + c\int_0^c \left(-\frac{x\,dx}{c-x}\right)$

But $-\frac{x}{c-x} = 1 - \frac{c}{c-x}$.

$\therefore n''c^2 = c^2 \log c - c^2 \log(c-c) + c\left[\int_0^c dx - \int_0^c \frac{c\,dx}{c-x}\right]$

$= c^2 \log. c - c^2 \log(c-c) + c^2 + c^2 \log.(c-c) - c^2 \log c = c^2$

$\therefore n'' = 1, \quad n''' = m n'' = \frac{1}{2}$ since $m = \frac{1}{2}$

12° Supported at both ends & loaded in the middle

$h^2 : h_0^2 :: c-x : x$

Integrate as in 10° & we find

$m'' = 2. \quad m''' = 1$

$n'' = 2/3 \quad - \quad n''' = \frac{1}{3}.$

13° Supported at both ends uniformly loaded.

$$h^2 : h_0^2 :: c^2 - x^2 : c^2 \quad \therefore \frac{h_0}{h} = \frac{c}{\sqrt{c^2-x^2}}$$

$$m''c = \int_0^c \frac{c\,dx}{\sqrt{c^2-x^2}} \cdot m'' = \int_0^c \frac{dx}{\sqrt{c^2-x^2}} = \sin^{-1}\frac{x}{c}.$$

$$= \sin^{-1}(1) \text{ when } x = c \ \& = 0 \text{ when } x = 0$$

$$\therefore m'' = \tfrac{1}{2}\pi = 1.5708$$

$$n''c = \int_0^c\int_0^x \frac{dx^2}{\sqrt{c^2-x^2}} = \int_0^c \sin^{-1}\frac{x}{c}.dx + a$$

$$= x\sin^{-1}\frac{x}{c} - \int_0^c \frac{x\,dx}{\sqrt{c^2-x^2}} = x.\sin^{-1}\frac{x}{c} + (c^2-x^2)^{1/2} = \tfrac{1}{2}\pi c - c$$

$$\therefore n'' = \tfrac{1}{2}\pi - 1 = 0.5708$$

— " —

C. E. Art. 176.

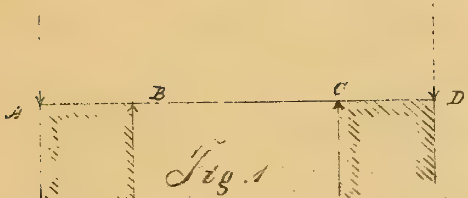


Fig. 1

The fixing of the ends of a beam is equivalent to applying two equal & opposite couples to the two ends as in Fig. 1. The mom^t thus produced bet B & C is constant (Case X. Art. 161.) This mom^t $(-M_1)$, as it acts upwardly, has to be subtracted from that due to the wt. on the beam. The slope due to this constant mom^t is

$$i_1 = \int \frac{M_1}{EI}dx = \frac{M_1 . c}{EI} \quad \text{whence}$$

$$M_1 = \frac{EI i_1}{c}$$

Ex. III. Eq.(9). From eq. 4). page. 248. C.E., $M = \frac{w}{2}(c^2-x^2)$ (w' being $= 0$) – But M_1 above $= \frac{1}{6}Wc = \frac{1}{6}w2c = \frac{1}{3}wc^2$

$\therefore M = M_1$ becomes

$$\frac{w}{2}(c^2-x^2) = \frac{1}{3}wc^2 \quad \therefore x = c\sqrt{\tfrac{1}{3}}$$

Ex. IV.

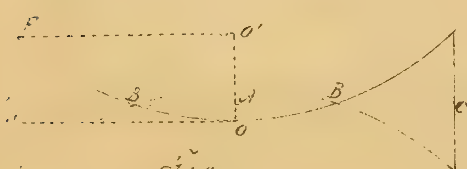


Fig..

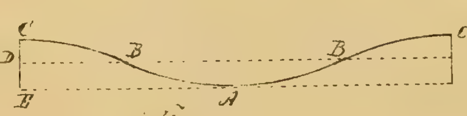


Fig. x

Since the beam is of uniform depth & of uniform strength, the curvature must be uniform in amt., for the outside fibre is subjected to the

same strain throughout. Hence the beam will be as in Fig. 2 where CB & BA & AB & BC are equal circular arcs.

The proof deflection at A (= CE) is evidently double that of a beam BAB of half the length merely supported but not fixed at the ends, the deflection in this last being = DE. But the deflection of a free beam of the length CC (= 2BB) would = 4 times that of BAB (= 4DE) since the deflection (Eq. 13) page 273) varies as the square of the length. Hence the deflection CE of the fixed beam of the same length = 2DE = $\frac{1}{2}$ that of the free beam.

In a free beam of uniform depth & strength $n'' = \frac{1}{2}$ in the expression for v_1. Hence in the fixed beam

$$v_1 = \frac{1}{4}\frac{fc^2}{Em'h}.$$

Considering BAB as a free beam, the mom.t of flexure at A = $M_0' = \frac{1}{8}wc^2$ (c being = BB). But $w = \frac{W}{2c}$

$$\therefore M_0' = \frac{1}{16}Wc = \frac{1}{32}Wl = \frac{M_0}{4}$$

Consequently the mom.t of flexure at C is

$$M_1 = \frac{3}{4}M_0 = \frac{3}{32}Wl$$

Let M' = actual mom.t at any point. Then, if M = mom.t of a free beam at same point & M_1 = mom.t at abutment

$$M' = M - M_1$$

$$\therefore M' = \frac{w(c^2 - x^2)}{2} - \frac{3}{16}Wc = \frac{w(c^2-x^2)}{2} - \frac{3}{8}wc^2 = \frac{wc^2}{8} - \frac{wx^2}{2}.$$

This is a parabola with vertex at a distance = $\frac{wc^2}{8}$ above A (Fig. 1.). Hence $\frac{wc^2}{8} = \frac{b_0}{2}$. At C, $x = c$ & $y_1 = \frac{3}{8}wc^2 = \frac{1}{2}b_1$ or breadth = 3 times that at A.

At any other point

$$y' : y_1 :: \left[\frac{wc^2}{8} - \frac{wx^2}{2}\right] : \frac{3}{8}wc^2$$

$$y' : \frac{b_1}{2} :: \quad " \quad : \quad "$$

$$\therefore y' = \left[\frac{1}{3} - \frac{4}{3}\cdot\frac{x^2}{c^2}\right]\frac{b_1}{2}.$$ Hence &c &c.

C. E. Art. 179.

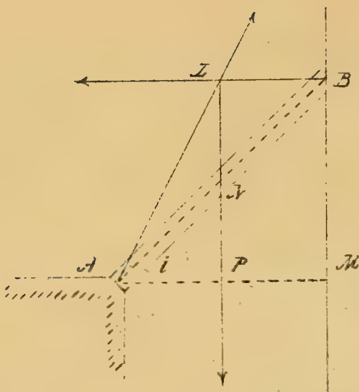


Fig. 1

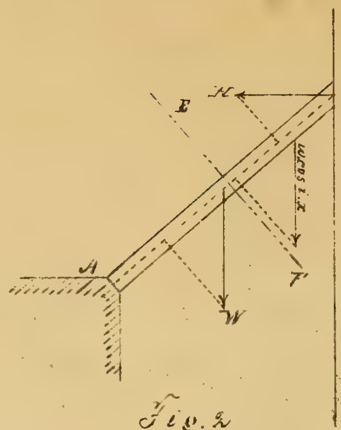


Fig. 2

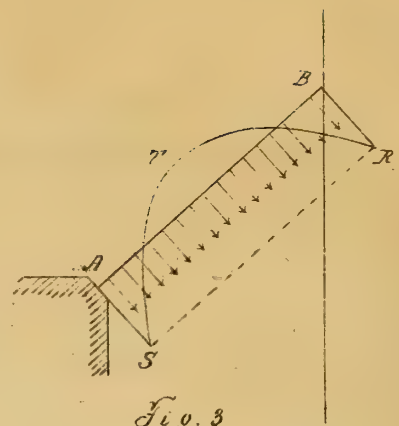


Fig. 3

On a sloping rafter uniformly loaded & supported at A & B (Fig. 1.) there are three external forces which keep the whole rafter in equilibrium. viz – the reactions of the two walls A & B & the resultant of the load on the whole beam.

This resultant acting vertically thro. the C. of G. (N) & the reaction of B, which is perp. to MB when the friction is left out of consideration, intersect at L. The reaction at A must be in the direction AL, since the directions of three balanced forces meet in one point. Hence ALP is the triangle of forces & since $LP = 2NP$

$$LP = W = 2NP = 2AP \tan i = 2H \tan i \quad \therefore \quad H = \frac{W}{2 \tan i}$$

Let the reaction at $A = T$; then

$$T = AL = \sqrt{W^2 + H^2}.$$

These forces produce a two-fold effect on AB. 1° A longitudinal compn. due to their compts. along the beam. – 2° A bending momt. due to the transverse compt. of the load.

1° Longitudinal Compression.

Let the intensity of the load measured along the horizontal $= w$. Then, if $AB = l$, $l \cos i =$ hor. length of the beam & $wl \cos i = W$ $\therefore$ $w = \frac{W}{l \cos i}$ & the intensity counting along AB is

$$\frac{W}{l} = w \cos i.$$

The longitudinal compt. at any section EF (Fig. 2) is that due to H plus that due to the load from B to EF (which

distance place $= x$); that is, it $= \mathcal{H}\cos i + wx.\cos i.\sin i$

$$= \frac{\mathcal{W}}{2\tan i}.\cos i + wx.\cos i.\sin i = \frac{\mathcal{W}}{2}.\frac{\cos^2 i}{\sin i} + wx\cos i\sin i$$

At A, this becomes

$$= \frac{\mathcal{W}}{2}.\frac{\cos^2 i}{\sin i} + wl\cos i\sin i = \frac{\mathcal{W}}{2}\left[\frac{1-\sin^2 i}{\sin i} + 2\sin i\right] = \frac{\mathcal{W}}{2}\left[\frac{1}{\sin i} + \sin i\right]$$

which, of course, = the reaction of the wall along AB.

If A = area of cross-section of beam, then the compn. per unit of surface is

$$p' = \frac{1}{A}\left[\frac{\mathcal{W}}{2}.\frac{\cos^2 i}{\sin i} + wx\cos i.\sin i\right] \quad \text{\& at } A$$

$$p' = \frac{\mathcal{W}}{2A}\left[\frac{1}{\sin i} + \sin i\right].$$

— " —

2° Bending Moment.

This is due to the transverse compt. of the load ($= \mathcal{W}\cos i$), the intensity of which is

$$\frac{\mathcal{W}\cos i}{l} = \frac{(wl\cos i)\cos i}{l} = w\cos^2 i$$

This momt. produces compn. in the upper fibres & tenn. in the lower ones if the beam is free. Let p'' = stress on outside fibre from this cause; then

$$M = \frac{p'' I}{m'h} \qquad \therefore p'' = \frac{M m'h}{I}$$

On top the quantities p' and p'' add & we should have $p'+p'' \leqq f'$ (= proof strength). At bottom the stress $= p'-p''$ which only becomes tenn. when $p'' > p'$.

The parabola STR (Fig. 3.) represents the momt. when the rafter is "fixed" at A & B. The max. momt. is then at A & B & $= \frac{2}{3}M_0$ (M_0 = momt. at centre of free beam). But $M_0 = \frac{1}{8}\mathcal{W}.\cos i.l \quad \therefore$ max. momt. $= M_1 = \frac{1}{12}\mathcal{W}.\cos i.l$

$$\therefore \quad p'' = \frac{1}{12}\,\frac{\mathcal{W}.l.\cos i.m'h}{I} \quad \text{which is at } A \,\&\, B$$

compression on bottom fibres of fixed beam. Since p' is a max. at A, the total max. stress is at A and =

$$p'+p'' = \frac{\mathcal{W}}{2A}\left[\frac{1}{\sin i} + \sin i\right] + \frac{\mathcal{W}l\cos i}{12}.\frac{m'h}{I} = \text{comp}^n \text{ on}$$

bottom fibres. Make $q = \frac{I}{m'h^2A}$ whence $m' = \frac{I}{qh^2A}$

$$\therefore p'+p'' = \frac{\mathcal{W}}{2A}\left[\frac{1}{\sin i} + \sin i + \frac{l\cos i}{6qh}\right].$$

C. E. Art. 180 -

(Insert following in place of whole article)

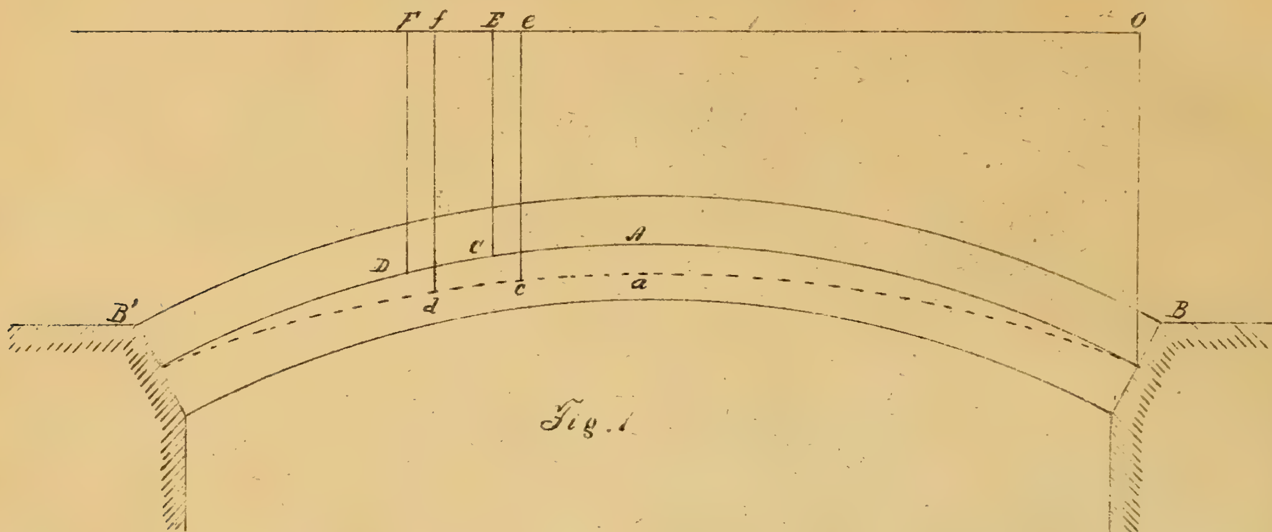


Fig. 1

Let O be the origin & x y the coordinates of any point on the neut. axis as c. Let l = span & k = rise. Let w = whole intensity of vert. load, whether constant or variable, per unit of hor distance, so that

$$\int_0^l w\,dx$$

= the whole load on the arch. The load $w\,dx$ on each element of the arch may be considered as made up of two parts

1° $w_1 dx$ producing direct compn. alone. This part of the load being distributed according to the laws of equilibrium of a linear arch (of the form which the neutral axis assumes after the load is on). From page 119 the eq. expressing the relation bet. this ~~force~~ part of the load & the arch is

$$\frac{dy}{dx} = \frac{P}{H} \text{ or } P = H \cdot \frac{dy}{dx}. \quad \text{But } w_1 = \frac{dP}{dx}$$

$\therefore w_1 = H \cdot \frac{d^2y}{dx^2}$ (H being the hor. thrust). The second part of the load is, of course,

$$(w - w_1)\,dx = \left(w - H\frac{d^2y}{dx^2}\right)dx \text{ -------- (1)}$$

& produces bending only. In any given case we substitute for w and $\frac{d^2y}{dx^2}$ their values & then proceed as follows -

The vert. compt of the shearing force at any point as c is due to the bending part of the load (the other part whose intensity = w_1 produces no shearing force)

& (page 242) it is

$$F = F_0 - \int_0^x (w - w_1)\, dx$$

$$\therefore \quad F = F_0 - \int_0^x w\, dx + \int_0^x H \cdot \frac{d^2y}{dx^2} : dx$$

$$\therefore \quad F = F_0 - \int_0^x w\, dx + H\left(\frac{dy}{dx} - \frac{dy_0}{dx_0}\right) \text{ ---------- } (2)$$

(F_0 being the still undetermined vertical component of the shearing force at B and $\frac{dy_0}{dx_0}$ the slope of the neut curve at that point).

The Bending Moment at C is (p. 243)

$$M = M_0 + \int_0^x F\, dx$$

$$= M_0 + F_0 x - \int_0^x \int_0^x w\, dx^2 + H \int_0^x \left(\frac{dy}{dx} - \frac{dy_0}{dx_0}\right) dx$$

$$= M_0 + F_0 x - \int_0^x \int_0^x w\, dx^2 - H\left[y_0 - y + x \frac{dy_0}{dx_0}\right] \text{ ----- } (3)$$

(M_0 being the still undetermined bending mom^t at B.).

The curvature produced in the arch by the part of the load that tends to bend it is

$$\frac{1}{r} = -\frac{M}{EI},$$

the (−) sign being used for the reason stated in the text; &, since the arch was already curved & the part of the load $\int w_1 dx$ does not affect the curv., the eq. $\frac{1}{r} = -\frac{M}{EI}$ expresses the alter-ation of curv. due to the load on the beam.

So also the alteration of slope is

$$i = \frac{dv}{dx} = \frac{ds}{r} = i_0 - \int \frac{M}{EI}\, ds$$

$$\therefore \quad i = i_0 - \int_0^x \frac{M}{EI} \sqrt{1 + \frac{dy^2}{dx^2}} \cdot dx \text{ -------------- } (4)$$

(i_0 being the still undetermined alteration of the slope at B).

The vert. deflection at C is

$$v = \int_0^x i\, dx \text{ -------------- } (5)$$

The four eq.^{s} (2) (3) (4) & (5) give the shearing force, bending mom^t, alteration of slope & deflection due to the bending action of the load. They contain four undetermined constants H F_0 M_0 and i_0. If in each of these eq.^{s}, x be made $= l$, then the resulting values of F

M i and v apply to the farther end of the span (at B′) & these particular values may be denoted by F, M, i, and v, respectively.

Let $ds = CD = \sqrt{dx^2 + dy^2}$ denote the length of an infinitely small part of the neut. curve. That arc is not altered in length by the bending action of the load, but it is altered by the direct compn (due to w_1) as follows –

The thrust along the rib due to the part $\int w_1 dx$ of the load is equal (p 199) to the hor. thrust mult.d by the secant of the inclination; that is, $= H \sec i$; &, if A = area of cross-section

$\frac{H \sec i}{A}$ = intensity of the thrust along the rib at C.

Let dt = the proportionate shortening of ds. Then

$$dt : ds :: \frac{H \sec i}{A} : E$$

$$\therefore \frac{dt}{ds} = \frac{H \sec i}{A.E} = -\frac{H . \frac{ds}{dx}}{E.A} \text{ - - - - - - - - } (6)$$

(The negative sign being inserted to denote compn)

To find the combined effect of the bending action & the compressive action on the figure of the neutral axis, proceed as follows;

Let u = the positive hor. displacement of a point in it, such as C. Let, for example, CD be the original position of an indefinitely small arc & cd its altered position. Let $OE = x$, $OF = x + dx$, $Oe = x + u$, $Of = x + u + dx + du$

$$EC = y, \quad FD = y + dy, \quad ec = y + v, \quad fd = y + v + dy + dv$$

$$CD = ds, \quad cd = ds + dt.$$

Then from the two eq.s

$ds^2 = dx^2 + dy^2$ and $(ds + dt)^2 = (dx + du)^2 + (dy + dv)^2$

we obtain by subtraction the following

$$2\,ds.dt + dt^2 = 2\,dx.du + du^2 + 2\,dy.dv + dv^2.$$

Reject dt^2, du^2 & dv^2 as of the second order when compared with $ds.dt$ &c, & we have

$$ds.dt = dx.du + dy.dv.$$

Hence the hor. displacement of D relatively to C is

$$du = \frac{ds}{dx} \cdot dt - \frac{dy}{dx} \cdot dv \text{ ----------- } (7)$$

From eq. (6) $\quad dt = -\dfrac{H \cdot \frac{ds}{dx} \cdot ds}{EA}$ and, since

$\dfrac{ds^2}{dx^2} = 1 + \dfrac{dy^2}{dx^2}$ and $dv = i\,dx$, we have

$$du = -\frac{H}{EA}\left[1 + \frac{dy^2}{dx^2}\right] ds - i\,\frac{dy}{dx} \cdot dx$$

$$\therefore \quad du = -\frac{H}{EA}\left[1 + \frac{dy^2}{dx^2}\right]^{\frac{3}{2}} dx - i\,\frac{dy}{dx} \cdot dx \text{ --------- } (7A)$$

which being integrated gives for the hor. displacement of C relatively to B & in a direction away from it

$$u = -\int_0^{x}\left[\frac{H}{EA}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} + i\,\frac{dy}{dx}\right] dx \text{ -------- } (8)$$

an expression containing the four undetermined constants already mentioned. If in the value of u we integrate bet zero & l, we obtain the alteration of the span BB′ which denote by u_1.

To find the undetermined constants F_0 M_0 i_0 H.

In the first place, the ends of the arched rib are either fixed or not fixed in position. If fixed, $i_0 = 0$. If free, $M_0 = 0$; so that in any practical case, the number of undetermined constants is reduced to three. Three eqs are needed to determine them.

If the abutments are immovable, $u_1 = 0$. If they yield, the value of u_1 may be found by experiment. Let it $= aH$. Then for a first eq. of condition we have

$$u_1 = 0 \text{ or } u_1 = aH \text{ --------------- } (9)$$

A second eq. of condition expresses the immobility in a vert. direction of B′ the farther end of the rib & is as follows.

$$v_1 = 0 \text{ --------------- } (10)$$

Again if the ends of the rib are fixed

$$i_1 = 0 \text{ --------------- } (11)$$

If free $\qquad M_1 = 0 \text{ --------------- } (11A)$

We can find the values of the three constants from eqs (9) (10) (11) or (11A) by elimination. These values introduced into eqs (3) & (5) render them available for calculation.

So $\int_0^2 i \cdot \frac{dy}{dx} \cdot dx = \int_0^2 \frac{dy}{dx} \cdot dv$. Integrate this last by the formula for parts

$$\int u dz = uz - \int z du. \text{ Let } u = \frac{dy}{dx} \text{ \& } dz = dv$$

then $z = v$ & $du = \frac{d^2y}{dx^2} \cdot dx$.

$$\therefore \int_0^2 \frac{dy}{dx} \cdot dv = \frac{dy}{dx} \cdot v_1 - \int_0^2 \frac{d^2y}{dx^2} \cdot v \cdot dx \ .$$

Since $v_0 = v_1 = 0$, this becomes

$$\int_0^2 \frac{dy}{dx} \cdot dv = -\int_0^2 \frac{d^2y}{dx^2} \cdot v dx = -\frac{8k}{l^2}\int_0^2 v dx \text{ ------ } (22)$$

This is simply proportional to the area of deflection $\int_0^2 v dx$.

Let w_0 = uniform fixed load per hor. unit & w = rolling load per hor. unit. Let the rolling load cover a hor. length = rl at the end farthest from O, the unloaded part being $(1-r)l$. Then eq.s (2)(3)(4) & (5) become as follows - denoting by (A) the formulae relating to the unloaded division & by (B) those relating to the loaded division.

-Shearing force.-

$$F = F_0 - \int_0^x w_0 dx + H\left[\frac{dy}{dx} - \frac{dy_0}{dx_0}\right]$$

$$\left.\begin{array}{ll} \text{(A) ------------} & = F_0 + \left[\frac{8kH}{l^2} - w_0\right] x \\ \text{and} & \\ \text{(B) -------} & F = F_0 + \left[\frac{8kH}{l^2} - w_0\right] x - w\left[x - (1-r)l\right] \end{array}\right\} \text{------ } (23)$$

since after we pass E going from the abutment at B, the shearing force is diminished constantly by the rolling load & at G, for instance, is less than at E by the load $w.EG = w(x-(1-r)l)$.

-Bending moment.-

$$M = M_0 + \int_0^x F dx$$

$$\left.\begin{array}{ll} \text{(A) ----------} & M = M_0 + F_0 x + \left[\frac{8kH}{l^2} - w_0\right]\frac{x^2}{2} \\ \text{and} & \\ \text{(B) -------} & M = M_0 + F_0 x + \left[\frac{8kH}{l^2} - w_0\right]\frac{x^2}{2} - \frac{w(x-(1-r)l)^2}{2} \end{array}\right\} \text{...} (24)$$

- Alteration of slope -

From (4A) $\quad i = i_0 - \frac{1}{EI_1}\int_0^x M\,dx$

But $i_0 = 0$

$$\therefore\ i = -\frac{1}{EI_1}\int_0^x M\,dx = -\frac{1}{qm'h^2A_1E}\cdot\int_0^x M\,dx$$

Substitute the value of M above & integrate & we have

$$\left.\begin{aligned}&(A)\text{-------}\ i = \frac{1}{qm'h^2A_1E}\left[-M_0x - F_0\frac{x^2}{2} - \left[\frac{8kh}{l^2} - w_0\right]\frac{x^3}{6}\right]\\ &\text{and}\\ &(B)\text{---}\ i = \frac{1}{qm'h^2A_1E}\left[-M_0x - F_0\frac{x^2}{2} - \left[\frac{8kh}{l^2} - w_0\right]\frac{x^3}{6} + \frac{w}{6}\left[x-(1-r)l\right]^3\right]\end{aligned}\right\}\text{..(25)}$$

- Deflection -

$$v = \int_0^x i\,dx$$

$$\left.\begin{aligned}&(A)\text{.....}\ v = \frac{1}{qm'h^2A_1E}\left[-M_0\frac{x^2}{2} - F_0\cdot\frac{x^3}{6} - \left[\frac{8kH}{l^2} - w_0\right]\frac{x^4}{24}\right]\\ &\text{and}\\ &(B)\ v = \frac{1}{qm'h^2A_1E}\left[-M_0\cdot\frac{x^2}{2} - F_0\frac{x^3}{6} - \left[\frac{8kH}{l^2} - w_0\right]\frac{x^4}{24} + \frac{w}{24}\left[x-(1-r)l\right]^4\right]\end{aligned}\right\}\text{.(26)}$$

The eq.^{s} of condition are the following -

(1) $\quad i_1 = 0$ or from (25)(B)

$$-M_0 - F_0\frac{l}{2} - \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{6} + \frac{wr^3l^2}{6} = 0 \text{ ---------- (27)}$$

By dividing out by l

(2) $\quad v_1 = 0$ or

$$-\frac{M_0}{2} - F_0\frac{l}{6} - \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{24} + \frac{wr^4l^2}{24} = 0 \text{ ----------- (28)}$$

The abutments being immovable,

(3) $\quad u_1 = 0$ or from (8A)

$$-\frac{H}{EA_1}\int_0^l\left(1 + \frac{dy^2}{dx^2}\right)dx - \int_0^l i\,\frac{dy}{dx}\cdot dx = 0 \quad \text{or}$$

$$-\frac{H}{EA_1}\left[l + \frac{16k^2}{3l}\right] + \frac{8k}{l^2}\int_0^l v\,dx = 0$$

Substituting for v its value from (26) and integrating

$$u_1 = 0 = -\frac{H}{EA_1}\left(l + \frac{16k^2}{3l}\right) + \frac{8k}{l^2}\cdot\frac{1}{qm'h^2EA_1}\left[-\frac{M_0l^3}{6} - \frac{F_0l^4}{24} - \left[\frac{8kH}{l^2} - w_0\right]\frac{l^5}{120} + \frac{w}{120}(rl)^5\right]$$

But $\int_0^l i_0 \cdot \frac{dy}{dx} \cdot dx = i_0(y_1 - y_0) = 0$. Hence substituting for I the value in (17).

$$u_1 = \frac{1}{EA_1}\left[-H\int_0^l \left(1+\frac{dy^2}{dx^2}\right)dx + \frac{1}{qm'h^2}\int_0^l \frac{dy}{dx}\int_0^x M\,dx^2\right] \text{------} (18)$$

In eq. (12) for the greatest stress -

1° Since the intensity of the direct stress of compⁿ is uniform, we have for that portion of the total stress

$$p' = \frac{H}{A_1}$$

2° Since the vert. deflection is the same at every point as in a uniform, straight, hor. beam with cross-section = A_1 & under same momᵗˢ as act on the arch, the stress in the outside fibres of the arched rib due to the bending part of the load must be the same as they would be in such a hor. beam; that is -

$$p'' = \frac{Mm'h}{I_1} = \frac{M}{qhA_1}$$

Hence

$$p_1 = \frac{1}{A_1}\left[H \pm \frac{M}{qh}\right] \text{------------} (12A)$$

from which eq. the area of cross-section may be readily obtained, if the depth & figure of the cross-section be assumed. The formulae of this Prob. give approximate results for flat, segmental ribs of uniform section.

Cor. The changes of temp. will change the hor. span & a term expressing this should be introduced into the value of u in (8A). Thus, if t = greatest probable deviation of temp. from the assumed standard & e = coefficient of expansion (See C.E. page. 539). Then the hor. expansion or contraction due to change of temp. is

$$\int_0^l t.e.\,dx = t.e.l.$$

Put this in (8A) and

$$u = -\frac{H}{EA_1}\int_0^x \left(1+\frac{dy^2}{dx^2}\right)dx \pm t.e.l - \int i\frac{dy}{dx}.dx \text{----------} (8A')$$

Prob. III presents no difficulty - See text.

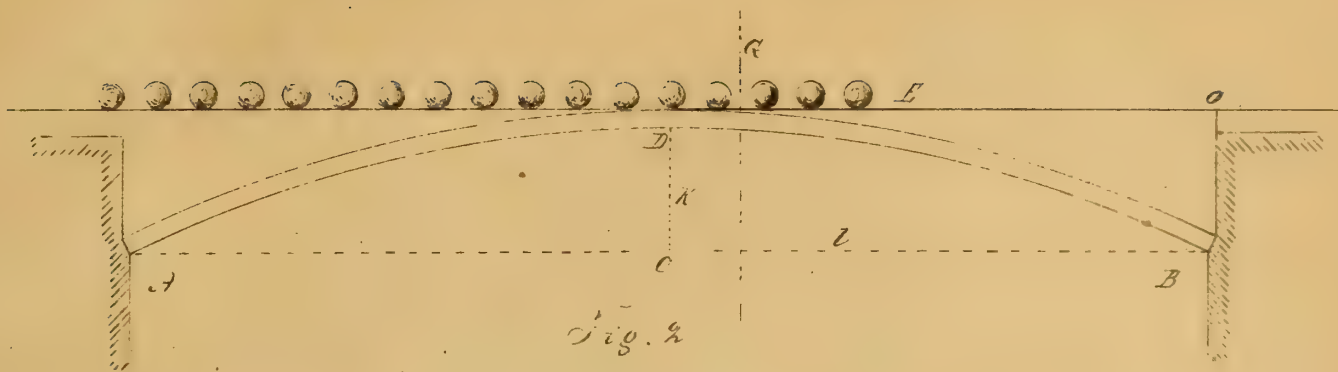


Fig. 2

Prob. IV.

Parabolic rib with rolling load; ends fixed in direction - abutments immovable. Most useful case is the rib of uniform depth & stiffness. If the ends are fixed,

$$\dot{i}_0 = \dot{i}_1 = 0 \text{ ---------- } (19)$$

Take the axis of x tang. to the neut. curve at its summit. ADB (Fig. 2.) = neut. curve; $CD = k$ = rise; $AB = l$ = span.

To transfer the origin from D to O. The eq. of the parabola referred to D as the origin, when DO is the axis of x & DC = axis of y, is $\quad x^2 = 2py$.

Eq.s of transformation are

$$y = y', \quad x = \frac{l}{2} - x' \quad \therefore \left(\frac{l}{2} - x'\right)^2 = 2py'$$

But from the eq. $x^2 = 2py$, when $x = \frac{l}{2}$, $y = k$

$$\therefore \frac{l^2}{4} = 2pk \quad \therefore 2p = \frac{l^2}{4k}. \qquad \text{Substituting}$$

this value in the last eq.

$$\frac{l^2}{4k} \cdot y' = \left(\frac{l}{2} - x'\right)^2 \quad \text{or dropping the primes}$$

$$y = \frac{4k}{l^2}\left(\frac{l}{2} - x\right)^2 \text{ ---------------- } (20)$$

From (20) by differentiation

$$\left.\begin{aligned} &\frac{dy}{dx} = -\frac{8k}{l^2}\left(\frac{l}{2} - x\right) \\ &\text{which, when } x = 0, \text{ is} \\ &\frac{dy_0}{dx_0} = -\frac{4k}{l} \\ &\text{Also} \quad 1 + \frac{dy^2}{dx^2} = 1 + \frac{64k^2}{l^4}\left(\frac{l}{2} - x\right)^2 \\ &\text{And} \quad \int_0^l \left(1 + \frac{dy^2}{dx^2}\right) dx = l + \frac{16k^2}{3l} \\ &\text{And} \quad \frac{d^2y}{dx^2} = \frac{8k}{l^2} \end{aligned}\right\} \text{ ------------- } (21)$$

Let thrust be considered as positive & tens.ⁿ as negative. The intensity of the stress at any cross-section due to the part of the load producing direct comp.ⁿ ($\int w_1 dx$) is

$$\frac{H \sec i}{A} = \frac{H \frac{ds}{dx}}{A} = p' \cdot (\text{page } 294)$$

while that on the outside fibre due to the bending part of the load is

$$\frac{M m'h}{I} = p'' \qquad (\text{page } 294)$$

Hence the greatest intensity of stress at any cross-section is

$$p_1 = \frac{H \cdot \frac{ds}{dx}}{A} \pm \frac{M m'h}{I} \text{ ---------------- } (12)$$

the positive or negative sign being used according as M acts towards or from the edge of the rib under consideration. The distance of the neut. axis from the edge of the rib $= m'h$.

From eq. (12) may be deduced the position of the point where the stress is a max. for a given arrangement of the load; the arrangement of the load that makes that stress an absolute max., & the corresponding value of the stress.

From eq. (4) (page 146. C.E) we see that the vertical deviation of the line of resistance from the neut. curve at any point is given by the expression

$$I_1 = \frac{M}{H} \text{ ------------------------- } (13)$$

& the normal deviation by $I_1' = \frac{M}{H \sec i} = \frac{M}{H \cdot \frac{ds}{dx}}$ ---------- (14)

These deviations take place in the direction in which M acts.

When the deflection is found by direct experiment, the greatest stress may be computed from it by the following: -

$$p_1 = \frac{H \frac{ds}{dx}}{A} \pm \frac{4 E m'h v}{n''l^2} \text{ ----------- } (15)$$

(The second term being obtained as in Art. 179 A. p. 296)

The preceding discussion applies to all cases of vert. loads.

— " —

Prob. II.

Rib of uniform stiffness. If the depth & figure of the cross-section of an arched rib are uniform

& its breadth is at each point proportional to the secant of the inclination of the rib at that point to horizon, that is, to

$$\frac{ds}{dx} = \sqrt{1+\frac{dy^2}{dx^2}}$$

then, if A_1 = sect. area at crown, the sect. area at any other point is

$$A = A_1 \sec i = A_1\sqrt{1+\frac{dy^2}{dx^2}} \text{ ----------- } (16)$$

So, since the mom.ts of inert. vary in similar figures as bh^3 & the $\underline{h}$ here is constant throughout the rib, if I_1 = mom.t of inert. at crown, then at any other point

$$I = I_1\sqrt{1+\frac{dy^2}{dx^2}} \text{ .... } (16)$$

Since both the direct thrust & area of cross-section vary as sec. i, the intensity of that thrust will be constant throughout the arch.

The alteration of slope (Eq. (4)) is

$$i = i_0 - \int_0^x \frac{M}{EI}\sqrt{1+\frac{dy^2}{dx^2}}\cdot dx$$

It becomes in this case

$$i = i_0 - \frac{1}{EI_1}\int_0^x M\,dx \text{ ------------ } (4A)$$

By referring to Art. 189, this expression will be found the same as that for the slope of a hor. straight beam of a uniform section = that of the arched rib at the crown & acted on by the same bending mom.ts. Hence the deflection ($v = \int i\,dx$) in both cases is the same.

Also eq. (6) in this case –

$$\frac{dC}{ds} = -\frac{H\frac{ds}{dx}}{EA} = -\frac{H}{EA_1} \text{ ------------ } (6A)$$

So, eq. (8) becomes

$$u = -\int_0^x\left[\frac{H}{EA}\left(1+\frac{dy^2}{dx^2}\right)^{3/2} + i\cdot\frac{dy}{dx}\right]dx$$

$$= -\frac{H}{EA_1}\int_0^x\left(1+\frac{dy^2}{dx^2}\right)dx - \int_0^x i\cdot\frac{dy}{dx}\cdot dy \text{ ----- } (8A)$$

For convenience we may express I in terms of the area & depth. From page 294 we have

$$I = qm'h^2A \text{ -------------- } (17)$$

Substituting in (8A) the value of i from eq. (4A) & integrating bet. 0 & l, we have

$$u_1 = \frac{H}{EA_1}\int_0^l\left(1+\frac{dy^2}{dx^2}\right)dx - \int_0^l i_0\cdot\frac{dy}{dx}\cdot dx + \frac{1}{EI_1}\int_0^l\frac{dy}{dx}\int_0^x M\,dx^2$$

Multiply this eq. by $\frac{9m'h^2A_1E}{8kl}$ and

$$-\frac{M_0}{6} - \frac{F_0 l}{24} + \frac{w_0 l^2}{120} + \frac{w r^5 l^2}{120} - H\left[\frac{k}{15} + \frac{9m'h^2}{8k}\left(1+\frac{16k^2}{3l^2}\right)\right] = 0 \dots (29)$$

Eliminate M_0 & F_0 from the eq. of condition to find H.
Multiply (28) by 2 & subtract it from (27). Then

$$-F_0\frac{l}{6} - \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{24} + \frac{wr^3l^2}{6} - \frac{wr^4l^2}{12} = 0 \dots\dots (\alpha)$$

Multiply (29) by 3 & subtract it from (28)

$$0 = -F_0\frac{l}{24} - \left[\frac{8kH}{l^2}-w_0\right]\frac{l^2}{24} + \frac{wr^4l^2}{24} - \frac{w_0l^2}{40} - \frac{wr^5l^2}{40} + H\left[\frac{k}{5} + \frac{3\,9m'h^2}{8k}\left(1+\frac{16k^2}{3l^2}\right)\right]$$

Multiply this last by 4 & subtract it from the preceding

$$\left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{12} + \frac{wr^3l^2}{6} - \frac{wr^4l^2}{4} + \frac{wr^5l^2}{10} + \frac{w_0l^2}{10} - 4H\left[\frac{k}{5} + \frac{3\,9m'h^2}{8k}\left(1+\frac{16k^2}{3l^2}\right)\right] = 0$$

Collecting the terms containing H

$$H\left[\frac{2}{3}k - \frac{4}{5}k - \frac{3\,9m'h^2}{2k}\left(1+\frac{16k^2}{3l^2}\right)\right] = -\frac{2w_0l^2}{120} - wl^2\left(\frac{r^3}{6} - \frac{r^4}{4} + \frac{r^5}{10}\right)$$

Multiply throughout by $\frac{15}{2}$ and

$$H\left[k + \frac{45\,9m'h^2}{4k}\left(1+\frac{16k^2}{3l^2}\right)\right] = l^2\left[\frac{w_0}{8} + w\left(\frac{15}{12}r^3 - \frac{15}{8}r^4 + \frac{15}{20}r^5\right)\right]$$

Let

$$\frac{45\,9m'h^2}{4k^2}\left(1+\frac{16k^2}{3l^2}\right) = B. \quad \text{Then}$$

$$H(1+B)k = \frac{l^2}{8}\left[w_0 + w(10r^3-15r^4+6r^5)\right] \quad \text{and}$$

$$H = \frac{l^2}{8(1+B)k}\left[w_0 + w(10r^3 - 15r^4+6r^5)\right] \dots\dots (31)$$

Multiply (28) by 3 & subtract it from (27) and

$$\frac{M_0}{2} - \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{24} + \frac{wr^3l^2}{6} - \frac{wr^4l^2}{8} = 0 \quad , \text{or}$$

$$M_0 = -\frac{w_0l^2}{12} - wl^2\left[\frac{r^3}{3} - \frac{r^4}{4}\right] + \frac{2}{3}kH. \quad \text{Substituting}$$

the value of H

$$M_0 = -\frac{w_0l^2}{12} - wl^2\left(\frac{r^3}{3}-\frac{r^4}{4}\right) + \frac{1}{12}\frac{l^2}{1+B}\left[w_0 + w(10r^3-15r^4+6r^5)\right]$$

$$\therefore\ -M_0 = \frac{w_0l^2}{12}\cdot\frac{B}{1+B} + \frac{wl^2}{12}\left[4r^3 - 3r^4 - \frac{10r^3-15r^4+6r^5}{1+B}\right] \dots (32)$$

From eq: (24) $M_1 = M_0 + F_0 l + \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{2} - \frac{wr^2l^2}{2}$:

But from (α)

$$F_0 l = w r^3 l^2 - \frac{w r^4 l^2}{2} - \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{2}$$

$$\therefore F_0 = wl\left[r^3 - \frac{r^4}{2}\right] - \left[\frac{8kH}{l^2} - w_0\right]\frac{l}{2}$$

Hence

$$M_1 = -\frac{w_0 l^2}{12}\cdot\frac{B}{1+B} - \frac{wl^2}{12}\left[4r^3 - 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right] + \frac{wl^2}{12}\left[12r^3 - 6r^4\right]$$
$$- \frac{l^2}{2}\left[\frac{8kH}{l^2} - w_0\right] + \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{2} - \frac{wl^2}{12}\cdot 6r^2 .$$

$$\therefore -M_1 = \frac{w_0 l^2}{12}\cdot\frac{B}{1+B} + \frac{wl^2}{12}\left[6r^2 - 8r^3 + 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right] \text{ - - - - - - -(33)}$$

Since the rib is fixed at the ends (from Art. 176) the greatest bending mom^t due to the part of the load producing a bending mom^t & consequently the greatest value of p'' & consequently the greatest value of p_1 occurs at the extremity of the loaded end. This value of p_1 is

$$p_1 = \frac{1}{A_1}\left[H + \frac{M_1}{qh}\right] .$$

Substitute for H & M_1 and

$$p_1 = \frac{1}{A_1}\left\{\frac{l^2}{8(1+B)k}\left[w_0 + w(10r^3 - 15r^4 + 6r^5)\right] + \frac{w_0 l^2}{12qh}\cdot\frac{B}{1+B}\right.$$
$$\left. + \frac{wl^2}{12qh}\left[6r^2 - 8r^3 + 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right]\right\} =$$

$$\frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{1}{k} + \frac{2}{3}\cdot\frac{B}{qh}\right) + w\left[\frac{2}{3qh}\left(6r^2 - 8r^3 + 3r^4\right) - \left(\frac{2}{3qh} - \frac{1}{k}\right)\frac{10r^3 - 15r^4 + 6r^5}{1+B}\right]\right\} \text{ - - -(34)}$$

In tension, let p_1' denote the greatest stress & q' the value of the factor q. Then, by an entirely similar reduction

$$p_1' = \frac{1}{A_1}\left[\frac{M_1}{q'h} - H\right]$$

$$\therefore p_1' = \frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{2B}{3q'h} - \frac{1}{k}\right) + w\left[\frac{2}{3q'h}\left(6r^2 - 8r^3 + 3r^4\right) - \left(\frac{2}{3q'h} + \frac{1}{k}\right)\frac{10r^3 - 15r^4 + 6r^5}{1+B}\right]\right\} \text{(35)}$$

Let r_1 = value of r corresponding to the absolute max. of thrust. Apply to eq (34) the test for a max; that is,

$\frac{dp_1}{dr} = 0$, or omitting the constant

factors,

$$\frac{2}{3qh}\left[12r-24r^2+12r^3\right]-\frac{30r^2-60r^3+30r^4}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right)=0$$

$$\therefore \frac{8}{qh}-\frac{16r}{qh}+\frac{8r^2}{qh}-\frac{30r}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right)+\frac{60r^2}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right)-\frac{30r^3}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right)=0$$

$$\therefore \frac{8}{qh}-r\left[\frac{16}{qh}+\frac{20}{(1+B)qh}-\frac{30}{(1+B)k}\right]+r^2\left[\frac{8}{qh}+\frac{40}{(1+B)qh}-\frac{60}{(1+B)k}\right]-r^3\left[\frac{20}{(1+B)qh}-\frac{30}{(1+B)k}\right]=0$$

$$\therefore \frac{8(1+B)k}{qhk(1+B)}-r\left[\frac{16(1+B)k+20k-30qh}{(1+B)qhk}\right]$$

$$+r^2\left[\frac{8(1+B)k+40k-60qh}{(1+B)qhk}\right]-r^3\left[\frac{20k-30qh}{(1+B)qhk}\right]=0$$

$$\therefore r^3(20k-30qh)-r^2(48k+8Bk-60qh)+r(36k+16Bk-30qh)=8(1+B)k$$

$$\therefore r^3-r^2\left[\frac{24k+4Bk-30qh}{10k-15qh}\right]+r\left[\frac{18k+8Bk-15qh}{10k-15qh}\right]=\frac{4(1+B)k}{10k-15qh}$$

$$\therefore r^3-r^2\left[2+\frac{4k+4Bk}{10k-15qh}\right]+r\left[1+2\left(\frac{4k+4Bk}{10k-15qh}\right)\right]-\frac{4k+4Bk}{10k-15qh}=0$$

This eq. may be factored thus -

$$(r^2-2r+1)\left(r-\frac{4k+4Bk}{10k-15qh}\right)=0. \quad \text{The roots are}$$

$$r=+1, \quad r=-1, \quad r=\frac{4(1+B)k}{10k-15qh}.$$

If we form $\frac{d^2p_1}{dr^2}$ & substitute in it the values of r, we find that $(r=+1)$ reduces it to zero. $(r=-1)$ to a quantity greater than zero! & consequently the value of r corresponding to a max. is

$$r_1=\frac{4(1+B)k}{10k-15qh}=\frac{2}{5}\left[\frac{1+B}{1-\frac{3}{2}\cdot\frac{qh}{k}}\right]$$

Similarly we find for tension ----- (36)

$$r_1'=\frac{2}{5}\cdot\frac{1+B}{1+\frac{3}{2}\cdot\frac{qh}{k}}$$

Substituting the value of r_1 for r in that member of p_1 which involves r, we have

$$\frac{2}{3qh}\left[6\frac{(4(1+B)k)^2}{(10k-15qh)^2}-8\frac{(4(1+B)k)^3}{(10k-15qh)^3}+3\frac{(4(1+B)k)^4}{(10k-15qh)^4}-10\frac{(4(1+B)k)^3}{(1+B)(10k-15qh)^3}\right.$$

$$+\frac{15(4(1+B)k)^4}{(1+B)(10k-15qh)^4} - \frac{6(4(1+B)k)^5}{(1+B)(10k-15qh)^5}$$

$$+\frac{1}{k}\cdot\frac{3qh}{2(1+B)}\left(\frac{10(4(1+B)k)^3}{(10k-15qh)^3} - \frac{15(4(1+B)k)^4}{(10k-15qh)^4} + \frac{6(4(1+B)k)^5}{(10k-15qh)^5}\right)\Bigg]$$

For brevity write $4(1+B)k = m$ & $(10k-15qh) = n$; then the above becomes

$$\frac{2}{3qh}\Big[6\frac{m^2}{n^2} - 8\frac{m^3}{n^3} + 3\frac{m^4}{n^4} - 10\frac{m^2\cdot 4k}{n^3} + 15\frac{m^3\cdot 4k}{n^4} - 6\frac{m^4\cdot 4k}{n^5}$$

$$+\frac{3qh\cdot 5\cdot m^2\cdot 4}{n^3} - \frac{15}{2}\cdot\frac{3qh\cdot m^3\cdot 4}{n^4} + 3\cdot\frac{3qh\cdot m^4\cdot 4}{n^5}\Big]$$

$$= \frac{2}{3qh}\Big[6\frac{m^2}{n^2} - 8\frac{m^3}{n^3} + 3\frac{m^4}{n^4} - 4\frac{m^2}{n^3}(10k-15qh)$$

$$+6\frac{m^3}{n^4}(10k-15qh) - \frac{12}{5}\cdot\frac{m^4}{n^5}(10k-15qh)\Big]$$

$$= \frac{2}{3qh}\Big[6\frac{m^2}{n^2} - 8\frac{m^3}{n^3} + 3\frac{m^4}{n^4} - 4\frac{m^2}{n^2} + 6\frac{m^3}{n^3} - \frac{12}{5}\cdot\frac{m^4}{n^4}\Big]$$

$$= \frac{2}{3qh}\Big[2\frac{m^2}{n^2} - 2\frac{m^3}{n^3} + \frac{3}{5}\cdot\frac{m^4}{n^4}\Big].$$ But $\frac{m}{n} = r_1$

hence

$$\text{max. } p_1 = \frac{l^2}{8A_1}\Big[\frac{w_0}{1+B}\Big(\frac{2B}{3qh} + \frac{1}{k}\Big) + \frac{2w}{3qh}\Big(2r_1^2 - 2r_1^3 + \frac{3}{5}r_1^4\Big)\Big] \text{----(37)}$$

In a way precisely analogous, we find

$$\text{max. tens.}^n\ p_1' = \frac{l^2}{8A_1}\Big[\text{same expression as above}\Big] \text{------(38)}$$

Eq.(37) serves to compute the proper sectional area when depth & form have been fixed.

If eq.(38) gives a negative result, there is no tension.

The value of F_0 may be obtained from (α) by substituting the value of H therein. Thus

$$-F_0 = \frac{l}{2}\Big[\frac{8kH}{l^2} - w_0\Big] - wr^3 l + \frac{wr^4 l}{2}$$

$$F_0 = \frac{l}{2}\Big[w_0 - \frac{1}{1+B}\big(w_0 + w(10r^3 - 15r^4 + 6r^5)\big) + 2wr^3 - wr^4\Big]$$

$$= \frac{l}{2}\Big[\frac{w_0 B}{1+B} + w\Big(2r^3 - r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\Big)\Big] \text{------(39)}$$

and this together with M_0 & H, substituted in (26) give us the deflection.

Since q is a fraction, if h the depth of the arch be small compared with k, its rise, we have $\frac{qh}{k}$ = a small fraction.

So in case of B the first factor or $\frac{45 q m' h^2}{4 k^2}$, if $\frac{h}{k}$ is small, will be very small, & the second or $1 + \frac{16 k^2}{3 l^2}$ will be less than 2 in most cases & when the arch is very flat may be but a little over 1. Then if we neglect $\frac{qh}{k}$, $\frac{q'h}{k}$ and B, we have

$$r_1 = r_1' = 2/5 \text{ ------------- } (36\,A)$$

and
$$p_1 = \frac{l^2}{8A_1}\left[w_0\left(\frac{2B}{3qh} + \frac{1}{k}\right) + 0.138\frac{w}{qh}\right] \text{ ------ } (37\,A)$$

$$p_1' = \frac{l^2}{8A_1}\left[w_0\left(\frac{2B}{3q'h} - \frac{1}{k}\right) + 0.138\frac{w}{q'h}\right] \text{ ----- } (38\,A)$$

When
$$\frac{1+B}{1-\frac{3}{2}\cdot\frac{qh}{k}} \geq \frac{5}{2}, \text{ then } r_1 = 1 = r_1'$$

and the greatest intensity of thrust as well as of tens.^{n} takes place when the beam is loaded over its whole length. In this case

$$H = \frac{l^2(w_0 + w)}{8(1+B)k} \text{ ------------- } (31\,B)$$

$$-M_0 = -M_1 = \frac{l^2}{12}\frac{(w_0 + w)B}{1+B} \text{ ------- } (33\,B)$$

$$p_1 = \frac{l^2}{8A_1}\left[\frac{w_0}{1+B}\left(\frac{2B}{3qh} + \frac{1}{k}\right) + 0.4\frac{w}{qh}\right] \text{ ------ } (37\,B)$$

$$p_1' = \frac{l^2}{8A_1}\left[\frac{w_0}{1+B}\left(\frac{2B}{3q'h} - \frac{1}{k}\right) + 0.4\frac{w}{q'h}\right] \text{ ------- } (38\,B)$$

Corollary

To provide for the effects of temperature, we have from (corollary to last Prop.) (8 A') by substitution

$$u = -\frac{H}{EA_1}\left(l + \frac{16k^2}{3l}\right) \pm t.e.l + \frac{8k}{l^2}\int_0^l v\,dx.$$

$$= -\frac{H}{EA_1}\left[l + \frac{16k^2}{3l} \mp \frac{t.e.l.A_1E}{H}\right] + x = 0$$

Multiply by $\frac{9m'h^2EA_1}{8kl}$ and

$$u = 0 = -H\left[\frac{k}{15} + \frac{9m'h^2}{8k}\left(1 + \frac{16k^2}{3l^2} \mp \frac{t.e.E}{p_0}\right)\right] + x$$

In this make

$$B = \frac{45\,qm'h^2}{4k^2}\left(1 + \frac{16k^2}{3l^2} \mp \frac{t.e.E}{p_0}\right)$$

(p_0 being the mean intensity of the thrust at the crown)

Use this value of B in the formulae of Prob. IV, & the results will include the effects due to change of temperature (See C.E. page 539.)

Prob. V.

If $u_1 = aH$, the expression from which (29) is deduced will have the form

$$-\left(l + \frac{16k^2}{3l} + aEA_1\right)\frac{H}{EA_1} + 8\frac{k}{l^2}\int_0^2 v\,dx = 0$$

and this can be reduced to a form similar to (29)

The value for H deduced from the eq.s of condition, it will be found, may be brought into the same form as (31) if B be made

$$= \frac{45\,qm'h^2}{4k^2}\left[1 + \frac{16k^2}{3l^2} + \frac{aEA_1}{l}\right].$$

To provide for change of temp, insert, as in the last case, in the second factor of the value of B a term $\mp \frac{t.e.E}{p_0}$

Prob. VI

Take a rib similar to that in Prob. V, but with ends not fixed. In this case $M_1 = M_0 = 0$ and i_0 = an undetermined constant. The action of the bending mom.t in this case is expressed as follows –

$$\left.\begin{array}{ll}(A) \text{------} & \mathcal{F} = \mathcal{F}_0 + \left[\frac{8k\mathcal{H}}{l^2} - w_0\right]x \\ (B) \text{------} & \mathcal{F} = \mathcal{F}_0 + \left[\frac{8k\mathcal{H}}{l^2} - w_0\right]x - w(x-(1-r)l)\end{array}\right\} \text{------------} (41)$$

$$\left.\begin{array}{ll}(A) \text{-----} & \mathcal{M} = \mathcal{F}_0 x + \left[\frac{8k\mathcal{H}}{l^2} - w_0\right]\frac{x^2}{2} \\ (B) \text{------} & \mathcal{M} = \mathcal{F}_0 x + \left[\frac{8k\mathcal{H}}{l^2} - w_0\right]\frac{x^2}{2} - w\frac{(x-(1-r)l)^2}{2}\end{array}\right\} \text{--------} (42)$$

$$\left.\begin{array}{ll}(A) \text{-----} & i = i_0 - \frac{1}{qm'h^2EA_1}\left[\mathcal{F}_0\cdot\frac{x^2}{2} + \left(\frac{8k\mathcal{H}}{l^2} - w_0\right)\frac{x^3}{6}\right] \\ (B) \text{-----} & i = i_0 - \frac{1}{qm'h^2EA_1}\left[\mathcal{F}_0\frac{x^2}{2} + \left(\frac{8k\mathcal{H}}{l^2} - w_0\right)\frac{x^3}{6} - \frac{w}{6}(x-(1-r)l)^3\right]\end{array}\right\} \text{---} (43)$$

$$\left.\begin{array}{ll}(A) \text{---} & v = i_0 x - \frac{1}{qm'h^2EA_1}\left[\mathcal{F}_0\cdot\frac{x^3}{6} + \left(\frac{8k\mathcal{H}}{l^2} - w_0\right)\frac{x^4}{24}\right] \\ (B) \text{---} & v = i_0 x - \frac{1}{qm'h^2EA_1}\left[\mathcal{F}_0\cdot\frac{x^3}{6} + \left(\frac{8k\mathcal{H}}{l^2} - w_0\right)\frac{x^4}{24} - \frac{w}{24}(x-(1-r)l)^4\right]\end{array}\right\} \text{---} (44)$$

If $w_1 = a\mathcal{H}$, as in Prob. V, we have for the first eq. of condition

$$0 = -\left(1 + \frac{16k^2}{3l^2} + \frac{aEA_1}{l}\right)\frac{l\mathcal{H}}{EA_1} + \frac{8k}{l^2}\int_0^l v\,dx \text{ --------- } (45)$$

Substitute the value of $\underline{v}$ from (44) & integrate. Then

$$0 = -\frac{\mathcal{H}l}{EA_1}\left[1 + \frac{16k^2}{3l^2} + \frac{aEA_1}{l}\right] + \frac{8k}{l^2}\left\{\frac{i_0 l^2}{2} - \frac{1}{qm'h^2EA_1}\left[\mathcal{F}_0\frac{l^4}{24} + \left(\frac{8k\mathcal{H}}{l^2} - w_0\right)\frac{l^5}{120} - \frac{w}{120}r^5l^5\right]\right\}$$

Multiply by $\frac{qm'h^2EA_1}{8k}$ and we have

$$(1) \text{ --------- } \frac{qm'h^2EA_1}{2}\cdot i_0 - \frac{\mathcal{F}_0 l^2}{24} + \frac{w_0 l^3}{120} + \frac{wr^5l^3}{120}$$

$$- \mathcal{H}l\left[\frac{k}{15} + \frac{qm'h^2}{8k}\left(1 + \frac{16k^2}{3l^2} + \frac{aEA_1}{l}\right)\right] = 0 \text{ ------- } (46)$$

The next eq. of condition $v_1 = 0$ is derived from (44), &, multiplying it by $\frac{qm'h^2EA_1}{2}$, we have

(2) ---- $q m' h^2 E A_1 i_0 - \frac{F_0 l^2}{6} + \frac{w_0 l^3}{24} + \frac{w r^4 l^3}{24} - \frac{H l k}{3} = 0$ ------(47)

And the third $M_1 = 0$ from (42) (by dividing by l) gives

(3) ------- $F_0 - \frac{w_0 l}{2} - \frac{w r^2 l}{2} + \frac{4 k H}{l} = 0$ ----------(48)

To eliminate i_0 from (46), (47), multiply the former by $\frac{l}{2}$ & subtract it from the latter, & then dividing by l^2, we have

$$-\frac{F_0}{12} + \frac{w_0 l}{40} + \frac{w l}{120}(5r^4 - 2r^5) - \frac{H k}{l}\left[\frac{1}{5} - \frac{q m' h^2}{4k^2}\left(1 + \frac{16k^2}{3l^2} + a\frac{E A_1}{l}\right)\right] = 0 \text{ --(49)}$$

Multiply this eq. by 12 & add it to (48) we get rid of F_0; then

$$-\frac{w_0 l}{5} - \frac{w l}{10}(5r^2 - 5r^4 + 2r^5) + \frac{H k}{l}\left(\frac{8}{5} + \frac{3 q m' h^2}{k^2}\left(1 + \frac{16k^2}{3l^2} + a\frac{E A_1}{l}\right)\right) = 0 \text{ --(50)}$$

From this $$H = \frac{l}{k \cdot \frac{8}{5}\left[1 + \frac{15}{8} \cdot \frac{q m' h^2}{k^2}\left(1 + \frac{16k^2}{3l^2} + \frac{a E A_1}{l}\right)\right]}$$

$$\times \frac{l}{5}\left[w_0 + \frac{w}{2}(5r^2 - 5r^4 + 2r^5)\right]$$

Placing $\frac{15}{8} \cdot \frac{q m' h^2}{k^2}\left[1 + \frac{16k^2}{3l^2} + \frac{a E A_1}{l}\right] = C$ ----------(51)

$$\therefore H = \frac{l^2}{8k(1+C)} \cdot \left[w_0 + \frac{w}{2}(5r^2 - 5r^4 + 2r^5)\right] \text{ ----------(52)}$$

From (48)

$$F_0 = \frac{w_0 l}{2} + \frac{w r^2 l}{2} - \frac{4 k H}{l}$$, substitute for H

& reduce &

$$F_0 = \frac{l}{2}\left[\frac{w_0 C}{1+C} + w\left(r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right] \text{ ------(53)}$$

From (47)

$$q m' h^2 E A_1 i_0 = \frac{F_0 l^2}{6} - \frac{w_0 l^3}{24} - \frac{w r^4 l^3}{24} + \frac{H l k}{3}$$

Substitute and

$$q m' h^2 E A_1 i_0 = \frac{l^3}{24}\left[\frac{2 w_0 C}{1+C} + 2w\left(r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right]$$

$$-\frac{l^3}{24} w_0 - \frac{l^3}{24} w r^4 + \frac{l^3}{24}\left[\frac{w_0}{1+C} + \frac{w}{2} \cdot \frac{5r^2 - 5r^4 + 2r^5}{1+C}\right]$$

$$= \frac{l^3}{24}\left[\frac{w_0 C}{1+C} + w\left(2r^2 - r^4 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right].$$

$$i_0 = \frac{l^3}{24 g m' h^2 E A_1}\left[\frac{w_0 C}{1+C} + w\left(2r^2 - r^4 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right] \text{--------} (54)$$

The shearing force at the loaded end of the rib (with sign reversed) is from (41)

$$P = -F_1 = -F_0 + w_0 l + w r l - \frac{8kH}{l}$$, or, replacing F_0 by its value from (48)

$$P = \frac{w_0 l}{2} + \frac{wl}{2}(2r - r^2) - \frac{4kH}{l}$$, &, replacing H by its value from (52)

$$P = \frac{l}{2}\left[\frac{w_0 C}{1+C} + w\left(2r - r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right] \text{----} (55)$$

To avoid negative signs this is denoted by P.

—,,—

<u>To find the max. bending moment.</u>

$$\frac{dM}{dx} = 0 = F_0 + x\left(\frac{8kH}{l^2} - w_0\right) - w\left(x - (1-r)l\right) \text{ from (42).}$$

$$\therefore \quad x = \frac{F_0 + wl(1-r)}{w_0 + w - \frac{8kH}{l^2}} = \frac{-P + w_0 l + w r l - \frac{8kH}{l} + wl(1-r)}{w_0 + w - \frac{8kH}{l^2}}$$

Hence the distance from the loaded end

$$l - x = \frac{P}{w_0 + w - \frac{8kH}{l^2}} \text{------------} (56)$$

Substitute the value just found for x in (42) & we find the max. bending mom.t Thus putting $\left(w_0 + w - \frac{8kH}{l^2}\right) = s$ for convenience

$$M' = F_0\left(l - \frac{P}{s}\right) + \frac{1}{2}\left(\frac{8kH}{l^2} - w_0\right)\left(l - \frac{P}{s}\right)^2 - w\frac{\left(rl - \frac{P}{s}\right)^2}{2}$$

$$= F_0\left(l - \frac{P}{s}\right) + \frac{1}{2}\left(\frac{8kH}{l^2} - w_0\right)\left(l^2 - \frac{2Pl}{s} + \frac{P^2}{s^2}\right) - \frac{w}{2}\left(r^2l^2 - \frac{2rlP}{s} + \frac{P^2}{s^2}\right)$$

$$= F_0\left(l - \frac{P}{s}\right) + \left(\frac{8kH}{l^2} - w_0\right)\left(\frac{l^2}{2} - \frac{Pl}{s}\right) - \frac{w}{2}\left(r^2l^2 - \frac{2Prl}{s}\right) + \frac{1}{2}\left(\frac{8kH}{l^2} - w_0 - w\right)\frac{P^2}{s^2}$$

$$= F_0\left(l - \frac{P}{s}\right) + \left(\frac{8kH}{l^2} - w_0\right)\left(\frac{l^2}{2} - \frac{Pl}{s}\right) - \frac{w}{2}\left(r^2l^2 - \frac{2Prl}{s}\right) - \frac{1}{2}\frac{P^2}{s}$$

now

$$F_0 = \frac{w_0 l}{2} + \frac{w r^2 l}{2} - \frac{4kH}{l} \text{ from (48). Hence}$$

$$M' = \frac{w_0 l^2}{2} + \frac{w r^2 l^2}{2} - 4kH - \frac{w_0 l P}{2s} - \frac{w r^2 l P}{2s} + \frac{4kHP}{ls} + 4kH$$
$$- \frac{w_0 l^2}{2} - \frac{8kHP}{ls} + \frac{w_0 P l}{s} - \frac{w r^2 l^2}{2} + \frac{w r l P}{s} - \frac{1}{2}\cdot\frac{P^2}{s}.$$

$$\therefore M' = \frac{w_0 l P}{2s} + \frac{2wrlP}{2s} - \frac{wr^2lP}{2s} - \frac{8kHP}{2ls} - \frac{P^2}{2s}$$

$$= \frac{1}{2s}\left[\left(w_0 l + 2wrl - wr^2 l - \frac{8kH}{l}\right)P - P^2\right]$$

But $\quad w_0 l + 2wrl - wr^2 l - \frac{8kH}{l} = 2P \quad$ (from 55)

Hence

$$\left.\begin{aligned} M' &= \frac{1}{2s}\left(2P^2 - P^2\right) = \frac{P^2}{2\left(w_0 + w - \frac{8k}{l^2}H\right)} \\ \& \therefore M' &= \frac{P(l-x)}{2} \end{aligned}\right\} \text{-------} (57)$$

As in Case IV, the greatest stress will be

$$p_1 = \frac{1}{A_1}\left[H + \frac{M'}{qh}\right] \text{----------------} (58)$$

$$= \frac{1}{A_1}\left\{\frac{l^2}{8k(1+C)}\left[w_0 + \frac{w}{2}\left(5r^2 - 5r^4 + 2r^5\right)\right]\right.$$

$$\left. + \frac{l^2}{4qh^2\left(w_0 + w - \frac{8kH}{l^2}\right)}\left[\frac{w_0 C}{1+C} + w\left(2r - r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right]^2\right\}$$

The last expression still has H in the denominator while the numerator involves P^2, or, what is the same, r^{10}. Hence when we differentiate the fraction p_1 the numerator of which involves r^{10} & the denominator r^5, we will obtain terms involving r^{14} in the eq. $\frac{dp_1}{dr} = 0$. This eq. is too complex for use. The value $r = 1/2$, however, gives a close approximation to the absolute max. thrust as we find by trial.

Introducing this value into (52) &c, we find

$$H = \frac{l^2}{8k(1+C)}\left(w_0 + \frac{w}{2}\right) \text{----------------} (52A)$$

$$F_0 = \frac{l}{2}\left[\left(w_0 + \frac{w}{4}\right)\frac{C}{1+C} - \frac{w}{4(1+C)}\right] \text{ or}$$

$$= \frac{l}{2}\left[\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} - \frac{w}{4}\right] \text{----------------} (53A)$$

(by adding & subtracting r^2).

$$i_0 = \frac{l^3}{24qm'h^2EA_1}\left[\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} - \frac{w}{16}\right] \text{--------} (54A)$$

$$P = \frac{l}{2}\left[\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} + \frac{w}{4}\right] \text{----------------} (55A)$$

by adding & subtracting $2r^2$ from the quantity in brackets in the value of P (eq. 55)

$$l - x = \frac{\frac{l}{2}\left[(w_0 + \frac{w}{2})\frac{c}{1+c} + \frac{w}{4}\right]}{w_0 + w - \frac{w_0 + \frac{w}{2}}{1+c}}$$

$$= \frac{\frac{l}{4}\left[w + 4(w_0 + \frac{w}{2})\frac{c}{1+c}\right]}{2\left[\frac{\frac{w}{2} + (w_0 + w)c}{1+c}\right]}$$ and by performing

the operations indicated in the denominator

$$l - x = \frac{\frac{l}{4}\left[w + 4(w_0 + \frac{w}{2})\frac{c}{1+c}\right]}{w + 2(w_0 + \frac{w}{2})\frac{c}{1+c}} \text{ - - - - - - - - - - - - } (56A)$$

And

$$M' = \frac{P(l-x)}{2} = \frac{l^2}{64}\,\frac{\left[w + 4(w_0 + \frac{w}{2})\frac{c}{1+c}\right]^2}{w + 2(w_0 + \frac{w}{2})\frac{c}{1+c}} \text{ - - - - - - - } (57A)$$

(For numerical example, see text, page 312)

Cor. 1. If we suppose the arch hinged at the crown as well as at the abutments (which supposition is made in Case. 4. Art. 374. C.E) the formulae reduce as follows.

In this case, M at the crown = 0. We thus have a new condition & can dispense, if we choose, with one of the other eq.s Let the mom.t at the crown = M_c. Then from eq. (42)

$$M_c = F_0 \cdot \frac{l}{2} + \left[\frac{8kH}{l^2} - w_0\right]\frac{l^2}{8} - w\,\frac{(\frac{l}{2} - l + rl)^2}{2} = 0$$

$$\therefore \frac{M_c}{l} = 0 = \frac{F_0}{2} - \frac{w_0 l}{8} + \frac{Hk}{l} - \frac{w}{2}\left(r^2 l - rl + \frac{l}{4}\right).$$

But from eq. (48) we have

$$\frac{M_1}{2l} = 0 = \frac{F_0}{2} - \frac{w_0 l}{4} - \frac{wr^2 l}{4} + \frac{2Hk}{l}.$$

Placing the last two eq.s equal to each other & collecting terms, we find

$$H = \frac{l}{k}\left[\frac{w_0 l}{8} - wl\left(\frac{r^2}{4} - \frac{r}{2} + \frac{1}{8}\right)\right] \quad \text{or}$$

$$H = \frac{l^2}{k}\left[\frac{w_0}{8} - w\left(\frac{r^2}{4} - \frac{r}{2} + \frac{1}{8}\right)\right] \text{ - - - - - - - - - - - } (52X)$$

Again from eq. (48)

$$\mathcal{F}_0 = \frac{w_0 l}{2} + \frac{wr^2 l}{2} - \frac{4k\mathcal{H}}{l}.$$ Substitute value of $\mathcal{H}$ and

$$\mathcal{F}_0 = \frac{w_0 l}{2} + \frac{wr^2 l}{2} - \frac{w_0 l}{2} + wlr^2 - 2wlr + \tfrac{1}{2}wl$$

$$\therefore \quad \mathcal{F}_0 = wl\left[\tfrac{3}{2}r^2 - 2r + \tfrac{1}{2}\right] \text{ - - - - - - - - - - - (53X)}$$

So in eq.(55)

$$P = -\mathcal{F}_1 = -\mathcal{F}_0 + w_0 l + wrl - \frac{8k\mathcal{H}}{l}$$ by reduction.

$$\therefore \quad P = wl\left(\frac{r^2}{2} - r + \frac{1}{2}\right) \text{ - - - - - - - - - - - - - (55X)}$$

and in eq. (57) the max. mom.^{t} is

$$\mathcal{M}' = \frac{P^2}{2\left[w_0 + w - \frac{8k\mathcal{H}}{l^2}\right]}$$

$$\therefore \mathcal{M}' = \frac{w^2 l^2\left[\frac{1}{2}r^2 - r + \frac{1}{2}\right]^2}{4w - 8wr + 4wr^2} \text{ - - - - - - - - - - - - - - (57X)}$$

When $r = 1/2$

$$\mathcal{H} = \frac{l^2}{8k}\left(w_0 + \frac{w}{2}\right) \quad ; \quad \mathcal{F}_0 = -\tfrac{1}{8}wl$$

$$P = \tfrac{1}{8}wl \quad ; \quad \mathcal{M}' = \frac{wl^2}{64}$$

By comparing these ~~last~~ results with the values of $\mathcal{H}$, $\mathcal{F}_0$, P and $\mathcal{M}'$ given in eq.^{s} (52A), (53A), (55A) and (57A), we see that the latter will reduce to the former by making $C = 0$.

To compare the general values of $\mathcal{H}$ &c obtained by this method (using $\mathcal{M}_c = 0$ as one of the eq.^{s} of condition) with those obtained in (52) &c by using $u_1 = 0$.

If in (52) & (52X) we make $C = 0$, these two eq.^{s} only become identical when

$$\tfrac{1}{8}wr^5 - \tfrac{5}{16}wr^4 + \tfrac{5}{16}wr^2 = -\left[\frac{wr^2}{4} - \frac{wr}{2} + \frac{w}{8}\right]$$

or

$$\tfrac{1}{8}r^5 - \tfrac{5}{16}r^4 + \tfrac{9}{16}r^2 - \frac{r}{2} + \frac{1}{8} = 0$$

We see that $r = 1$ and $r = 1$ are two of the roots of this eq. and also the the other roots are

$$r = 1/2 \text{ and } r = \pm\sqrt{2}.$$

Hence the value $C = 0$ only renders the values of $\mathcal{H}$ &c identical when $r = 1$ or $r = 1/2$.

If we suppose the example given on page

312 of the text to be hinged at the crown, we find

$$H = 1.50\,l\,w$$

$$p_1 = \frac{1}{A_1}\left[\frac{120\,wl}{64} + 1.50\,wl\right]$$

$$= \frac{wl}{A_1}[1.875 + 1.50]$$

$$= \frac{wl}{A_1}[3.375]$$

When the rib is continuous at the crown in the example on page 312, $p_1 = \frac{wl}{A_1}(3.51)$. Hence the gain by hinging at the crown in this case is about 4%.

Cor. 2. The effects of change of temp. can be introduced into the eq.s of Prob. VI in a manner similar to that used in Prob. IV.

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Prob. VII.

To find the greatest deflection of an arched rib.

Evidently the greatest value of v must occur when $i = 0$; that is, when the alteration of slope after having gone thro. all its positive values becomes zero; for it is the integral of these values which constitutes v & the sum of all of them will = max. v. Find the values of x in (25) when $i = 0$ & substitute these values or the one of them which will evidently be applicable in (26); & then forming $\frac{dv}{dr} = 0$, find the value of r which will render v a max. This process gives eq.s of extreme intricacy & involving r to a high power, hence they are not available for use. But by making suppositions on them which will include the ordinary practical cases, it has been determined that the absolute max. deflection occurs at the middle of the rib & that to produce it the passing load should extend over the whole length. Hence when v is a max., $r = 1$ and $x = \frac{l}{2}$.

Using these values of r & x, eq. (31) becomes

$$\mathcal{H} = \frac{l^2(w+w_0)}{8k(1+B)}.$$

Eq. (39) becomes $\mathcal{F}_0 = \dfrac{l(w+w_0)B}{2(1+B)}$

Eq.s (32) & (33) " $-\mathcal{M}_0 = -\mathcal{M}_1 = \dfrac{l^2(w+w_0)B}{12(1+B)}$ These values with those of r & x being substituted in (26)(B) give

$$v = \frac{1}{gm'h^2EA_1}\left[\frac{l^2(w+w_0)B}{12(1+B)}\cdot\frac{l^2}{8} - \frac{l(w+w_0)B}{2(1+B)}\cdot\frac{l^3}{48} - \frac{w-Bw_0}{1+B}\cdot\frac{l^4}{384} + \frac{wl^4}{384}\right]$$

$$= \frac{1}{gm'h^2EA_1}\left[\frac{l^2(w+w_0)B}{12(1+B)}\cdot\frac{l^2}{8} - \frac{l(w+w_0)B}{2(1+B)}\cdot\frac{l^3}{48} - \left[w - \frac{(w+w_0)B}{1+B}\cdot\frac{l^4}{384}\right] + \frac{wl^4}{384}\right]$$

$$= \frac{1}{gm'h^2EA_1}\left[\frac{l^4(w+w_0)B}{96(1+B)} - \frac{l^4(w+w_0)B}{96(1+B)} + \frac{l^4(w+w_0)B}{384(1+B)}\right]$$

$$\therefore\quad v = \frac{l^4(w+w_0)B}{384gm'h^2EA_1(1+B)}.$$

If $c = \frac{l}{2}$ (as in Art. 169. C.E.) & $l(w+w_0) = W$, the above may be written

$$v = \frac{Wc^3}{48EI}\cdot\frac{B}{1+B}.$$

So too eq (52) becomes

$$\mathcal{H} = \frac{l^2(w+w_0)}{8k(1+C)}$$

Eq. (53) " $\mathcal{F}_0 = \dfrac{l(w+w_0)C}{2(1+C)}$

Eq. (54) " $i_0 = \dfrac{l^3(w+w_0)C}{24gm'h^2EA_1(1+C)}$

Substitute in (44)(B) and

$$v = \frac{l^4(w+w_0)C}{48gm'h^2EA_1(1+C)} - \left[\begin{array}{l}\text{a term precisely analogous}\\ \text{to 2nd member of eq. (26)(B)}\end{array}\right]$$

$$= \frac{6l^4(w+w_0)C}{384gm'h^2EA_1(1+C)} - \frac{l^4(w+w_0)C}{384gm'h^2EA_1(1+C)}$$

$$\therefore\quad v = \frac{5l^4(w+w_0)C}{384gm'h^2EA_1(1+C)} \quad\text{..............}\quad (62)$$

When $l = 2c$ & $W = (w + w_0)l$, this becomes

$$v = \frac{5c^3W}{48EI} \cdot \frac{C}{1+C} .$$

Now eq. (12) page 273 C.E. gives when we substitute for n''' from Case V. page 274

$$v_1 = \frac{5}{48} \cdot \frac{Wc^3}{EI}$$

Hence if I, W, c be the same in this as in (62)

$$v : v_1 :: \frac{C}{1+C} : 1 :: C : 1+C$$

When the straight beam is fixed at the ends, eq. (4) page 284. C.E. shows that the deflection is diminished in the ratio $n'' : (n'' - \frac{m''}{2})$ which in this case is

$$\frac{5}{12} : (\frac{5}{12} - \frac{4}{12})$$

$$5 : 1 .$$

Hence in such a beam,

$$v_1 = \frac{1}{48} \cdot \frac{Wc^3}{EI}$$

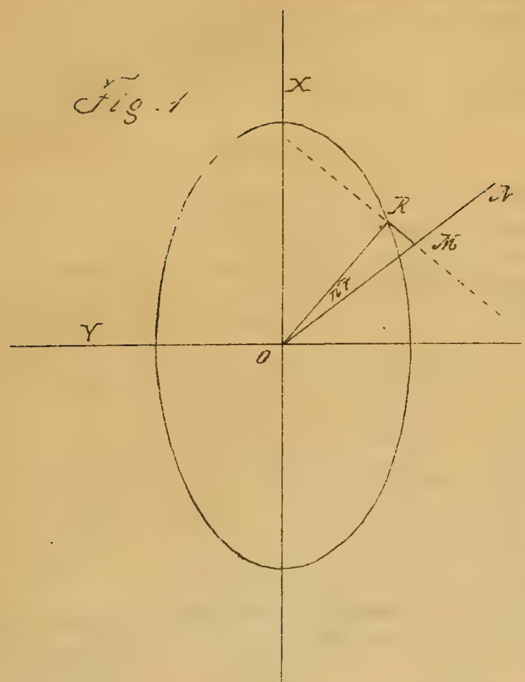
and this compared with (61) gives

$$v : v_1 :: \frac{B}{1+B} : 1 \text{ or } :: B : 1+B$$

For Prob. $\underline{VIII}$ – See text.

C. E. Art. 183.

Fig. 1



In place of this article, substitute arts. 194-5-6-7-8. in Rank.s A. M., adding there-to the table on page 324 of the C.E & the following notes.

For explanation of the eq.s in Art. 194. A. M (or those on page 319 C.E) see notes on ellipse of stress.

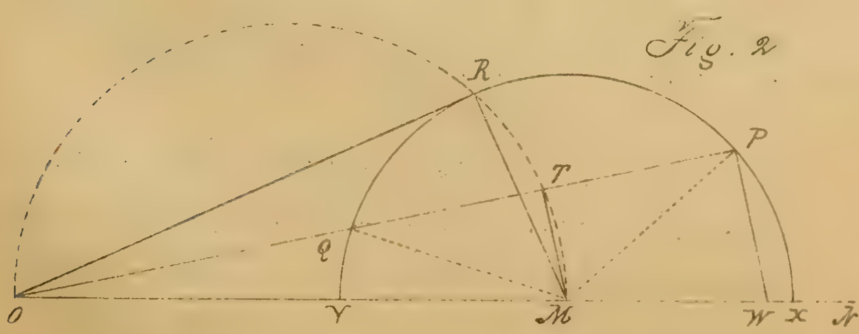
The geometrical construction of the ratios of the conj. pressures (page 320. C.E) can also be gotten from the discussion of the ellipse of stress. Thus

Fig. 1. illustrates the max. obliquity of the stress which takes place when MR is perp. to OR. Note that all the semi-diam.s of the ellipse (or all the values of OR) are but the resultants of the constant lines OM & MR, (OM being = the half-sum & MR = the half-difference of the "principal stresses").

Now, having given the angle of max. obliquity (NR in Fig. 1.) & also the actual obliquity of two conj. stresses, it is easy to get their ratio by the construction in Fig. 2. Thus - draw ON (Fig. 2.) indefinitely & OR, making an angle = φ = max. obliquity, with it. At any convenient point M, draw a perp. to OR (which is most readily done by means of the dotted circle ORM).

Then OM & MR will be to each other as the half-sum & half-difference of the principal stresses. Draw OP making an angle with ON = θ & evidently OQ & OP are proportional to the two semi-diam.s having

Fig. 2



the given obliquity, since they are both the resultants of OM & MR. Hence the ratio of the two conj. pressures is that of $\dfrac{OP}{OQ}$ or $\dfrac{OQ}{OP}$.

The analytical expression for this ratio is obtained in Prob. V (Art. 112. A. M.), but it may also be gotten from Fig. 2. Draw the chord MT. It is perp. to OT since the angle OTM is inscribed in a semi-circle. Therefore the chord QP is bisected at T or $QT = TP$. Also

$$OM \cos\theta = OT \text{ and } OM \cos\varphi = OR$$

$$OP = OT + TP = OT + QT$$

$$\overline{OR}^2 = \overline{OQ}.\overline{OP} = (OT - QT)(OT + QT) = \overline{OT}^2 - \overline{QT}^2$$

$$\therefore \quad QT = \sqrt{\overline{OT}^2 - \overline{OR}^2} = OM\sqrt{\cos^2\theta - \cos^2\varphi}.$$

Hence

$$OP = OM\left[\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}\right]$$

and

$$OQ = OT - TQ = OM\left[\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}\right]$$

$$\therefore \quad \frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}} = \frac{OP}{OQ} = \frac{p}{p'}$$

So too $\quad OX = OM + MX = OM + MR = OM(1 + \sin\varphi)$

$$OY = OM(1 - \sin\varphi)$$

$$\therefore \quad \frac{OX}{OY} = \frac{1 + \sin\varphi}{1 - \sin\varphi}.$$

Again, if PH be drawn perp. to OP, we have

$$OH : OP : OQ :: wx : p : p'$$

— " —

Eq.s (5) & (6) (page 217. A. M.) are obtained by substituting for p_x and p_y their values from (1) & (2) (page 216. A. M.).

In Fig. 56. (Art. 112. A. M.)

when the stresses are conj., OR will coincide with $A'B'$ (one of the planes of stress) & OR' will coincide with

AB (the other plane of stress), & since one of the conj. stresses in the earth is vert., ON which is the normal to AB (or OR′) will be hor. Also, since the stresses are conjugate, the obliquities are equal or $\hat{nr} = \hat{n'r'}$ (see Case. 3. Art. 112. A.M.)

Therefore eq. (17) (Art. 112. A.M.) gives

$$\cos 2\angle\hat{n} = \frac{2p_x \cos\theta - p_1 - p_2}{p_1 - p_2} = 2\psi$$

In Fig. 57. (Art. 112. A.M.) when the stresses are conjugate, the points O, R′, S and R are in one straight line to which MS is perp. Now

$2\angle\hat{n} = NMR = 2\psi$. But in the figure $NMR = NOR + MRO$.

NOR = common obliquity of the stresses $= \theta$.

$$MS = MR \sin MRO \quad \therefore \quad \sin MRO = \frac{MS}{MR} \text{. But}$$

$$MS = OM \sin\theta.$$

Draw $MR_1 = MR$ & perp. to OR_1. Then the angle $MOR_1 = \varphi$

$$MR_1 = MR = OM \sin\varphi$$

$$\therefore \quad \sin MRO = \frac{OM \sin\theta}{OM \sin\varphi} \quad \therefore \quad MRO = \sin^{-1}\left(\frac{\sin\theta}{\sin\varphi}\right)$$

$$\therefore \quad \angle\hat{n} = \psi = \tfrac{1}{2}\left[\theta + \sin^{-1}\left(\frac{\sin\theta}{\sin\varphi}\right)\right]$$

If we take NMR′ instead of NMR, we have

$$NMR' = NOR + MR'O$$

$$\text{But} \quad MR'O = 180° - MR'R = 180° - MRO$$

$$\therefore \quad NMR' = NOR + 180° - MRO$$

$$\therefore \quad \angle\hat{n} = \psi = \tfrac{1}{2}\left[\theta + 180° - \sin^{-1}\left(\frac{\sin\theta}{\sin\varphi}\right)\right]$$

—"—

C. E. Art. 186.

Case III. From eq. 2. we get

$$V = \frac{\frac{l}{2}}{3}\left[S_0 + 4S_1 + S_2\right]$$

$$S_0 = 2b_0 h_0 + s h_0^2 \quad , \qquad S_1 = 2b_0 \frac{h_0 + h_2}{2} + s\left[\frac{h_0 + h_2}{2}\right]^2.$$

$$S_2 = 2b_0 h_2 + s h_2^2.$$

$$V = \frac{\varkappa}{6}\left[2b_0h_0 + sh_0^2 + 4\left(2b_0\frac{h_0+h_2}{2} + s\left(\frac{h_0+h_2}{2}\right)^2\right) + 2b_0h_2 + sh_2^2\right]$$

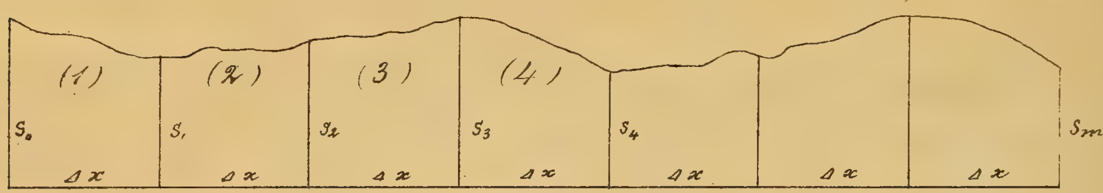
$$V = \frac{\varkappa}{6}\left[2b_0(h_0+h_2) + s(h_0^2+h_2^2) + 4b_0(h_0+h_2) + s(h_0+h_2)^2\right]$$

$$V = \varkappa\left[b_0(h_0+h_2) + s\cdot\frac{h_0^2+h_2^2}{6} + s\cdot\frac{h_0^2+2h_0h_2+h_2^2}{6}\right]$$

$$V = \varkappa\left[b_0(h_0+h_2) + \frac{s(h_0^2+h_0h_2+h_2^2)}{3}\right] \quad \text{- - - - - - - - - - - (3)}$$

$$\text{or } V = \varkappa\left[b_0(h_0+h_2) + s\left(\frac{(h_0+h_2)^2}{4} + \frac{(h_0-h_2)^2}{12}\right)\right] \quad \text{- - - - - - - - - - - (4)}$$

Case IV

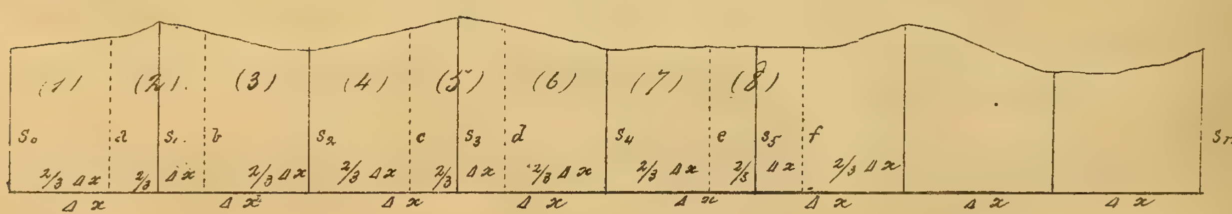


$$(1) = \Delta\varkappa\cdot\frac{S_0+S_1}{2}\ ;\ (2) = \Delta\varkappa\,\frac{S_1+S_2}{2}\ ;\ (3) = \Delta\varkappa\cdot\frac{S_2+S_3}{2}\ \ \&c$$

$$V = \Delta\varkappa\left[\frac{S_0+S_1}{2} + \frac{S_1+S_2}{2} + \frac{S_2+S_3}{2} + \frac{S_3+S_4}{2} + \&c\right]$$

$$V = \Delta\varkappa\left[\frac{S_0}{2} + S_1 + S_2 + S_3 + \ldots\ldots\ldots\ldots \frac{S_m}{2}\right] \ldots\ (5)$$

Case V



$$(1) = \tfrac{2}{3}\Delta\varkappa\cdot\frac{S_0+a}{2}\ ;\ (2) = \tfrac{2}{3}\Delta\varkappa\cdot\frac{a+b}{2}\ ;\ (3) = \tfrac{2}{3}\Delta\varkappa\cdot\frac{b+S_2}{2}\ ;\ (4) = \tfrac{2}{3}\Delta\varkappa\cdot\frac{S_2+c}{2}$$

$$V = \tfrac{2}{3}\Delta\varkappa\left[\frac{S_0+a}{2} + \frac{a+b}{2} + \frac{b+S_2}{2} + \frac{S_2+c}{2} + \frac{c+d}{2} + \frac{d+S_4}{2} + \&c\right]$$

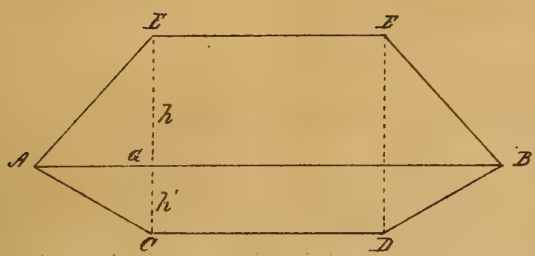
$$V = \tfrac{2}{3}\Delta\varkappa\left[\frac{S_0}{2} + \frac{2(a+b) + 2S_2 + 2(c+d) + 2S_4 + \&c}{2}\right]$$

But $\frac{a+b}{2} = S_1$ and $\frac{c+d}{2} = S_3$ &c

$$\therefore\ V = \frac{\Delta\varkappa}{3}\left[S_0 + 4S_1 + 2S_2 + 4S_3 + 2S_4 + \ldots\ldots\ldots S_n\right] \ldots\ (6)$$

C. E. Art. 204

To obtain Eq. 1.



Consider first the conditions of equilibrium in the earth AECDB before the excavation & filling. The max. hor. thrust in that earth consistent with stability is given by eq. (3A) Art 183. C.E. viz.

$\frac{p'}{p} = \frac{1+\sin\varphi}{1-\sin\varphi}$ where p' = hor. & p = vert. pressure.

$\therefore p' = p\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)$ is the superior limit of the intensity of the hor. thrust that may exist in the soft earth.

Now the embankment may be made with a limit of heaviness determined by this max. hor. thrust. Thus, eq. (1A) Art. 183 gives

$$\frac{p_1}{p_2} \leqq \frac{1+\sin\varphi}{1-\sin\varphi}.$$

In this p_2 = hor. & p_1 = vert. pressure & this eq. serves to give the greatest vert. pressure p_1 which is consistent with the hor. pressure p_2.

$$\therefore p_1 = p_2\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)$$

But, if we make the hor. pressure = p' above, we have

$$p_2 = p' = p\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)$$

$$\therefore p_1 = p\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2 = \frac{p}{k'^2}$$

Now $p_1 = w(h+h')$ being the vert. pressure due to the embankment, while $p = w'h'$ being the vert. pressure due to the earth in its natural state.

Both these refer to the bottom of the ditch along CD.

Substitute and

$$w(h+h') = w'\frac{h'}{k'^2}$$

$$\therefore whk'^2 + wh'k'^2 = w'h'$$

$$\therefore h' = \frac{whk'^2}{w' - wk'^2}$$

C. E. Art. 236

Eq. 1. δ being the distance from the neut. axis (in a direction perp. to it) to the centre of pressure, it is $= x_0$ in eq. 4. (Art. 94. A. M.)

$$\therefore \quad \delta = \frac{aI}{P}$$

This is the expression for the deviation in that direction of any uniformly varying stress.

Now, if we add the condition that the varying stress shall be $=0$ at one side or, what is the same, shall not exceed twice the mean stress at the other, we know from the value for the intensity of the stress (A. M. Art. 94)

$$p = p_0 + ax.$$

When this is applied to the point of max stress, that

$$p_1 = p_0 + ax_1 = 2p_0 \quad \text{------------} \quad (c)$$

But the mean stress $p_0 = \frac{P}{A}$. From (c) $a = \frac{p_1 - p_0}{x_1} = \frac{p_0}{x_1} = \frac{P}{Ax_1}$.

In the case before us this deviation x_1 is called y.

$$\therefore \; a = \frac{P}{Ay}.$$

Substitute in value of δ, and

$$\delta = \frac{I}{Ay}$$

From Art. 179. C. E, we find

$$q = \frac{I}{m'h^2A}$$

$$\therefore \; qh = \frac{I}{m'hA} = \frac{I}{Ay} \quad (\text{since } m'h = y)$$

$$\therefore \quad \delta = \frac{I}{Ay} = qh.$$

—"—

C. E. Art. 237.

Case I. We have already seen in the case of foundations for an embankment (Case III. Art. 204. C. E.) that the limit of the safe pressure of the structure is

$p_1 = p\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2$ where $p =$ the intensity of the pressure of the earth that has been displaced $= wx$. Here $p_1 = p_0$.

$$\therefore \; \frac{wx}{p_0} \geq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2$$

Case. II.

Of course, the limit of max. pressure is that just deduced. For the min. consider that the hor. pressure due to the normal condition of the earth is

$$p' = p\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$$

Now this pressure exists at the bottom of the foundation in the sides of the trench over which the earth still stands, & to keep these sides from being pushed in or from raising the bottom up the wt. of the foundation must be at least = the wt. of the earth taken out; for it will take that much wt. to develop a hor. pressure = the above & which will therefore keep it in equilibrium. The eq. (1A) (Art 183. C.E) gives

$$\frac{p_h}{p_2} \geqq \frac{1-\sin\varphi}{1+\sin\varphi}$$ where p_h = hor. & p_2 = vert. pressure.

$$\therefore\ p_2 \geqq p_h\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right).$$ But the hor. pressure p_h = that already in the earth = p'. $\therefore\ p_2 \geqq p (= wx)$.

$$\therefore\ \frac{wx}{p_2} \leqq 1.$$

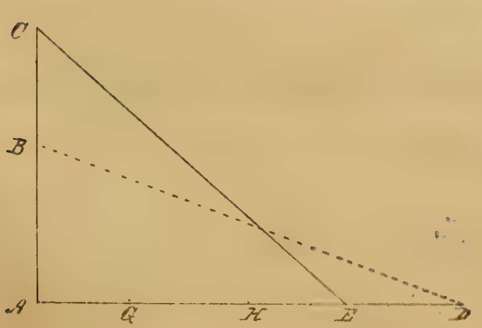
Eqs (2) & (3) combined give (4). Eq. 5. is deduced thus—
From eq. (1). (Art 236. C.E.)

$$\delta = \frac{aI}{P}.$$ If p_2 = least pressure on the foundation, we have the general value

$$p_2 = p_0 - ax_1 = p_0 - ay \text{ (as } y \text{ here is used for } x_1)$$

$$\therefore\ a = \frac{p_0 - p_2}{y} \quad \therefore \quad \delta = \frac{p_0-p_2}{y} \cdot \frac{q\,m'h^2A}{Ap_0} = \frac{p_0-p_2}{p_0} \cdot qh.$$

When $p_2 = 0$ or the max. pressure = twice the mean, this expression becomes = qh (as on page 278 C.E.)



" C. E. Art. 263.

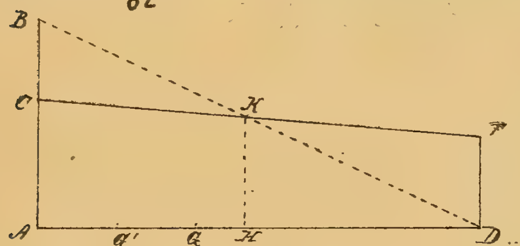
Let $AD = t$ = thickness of wall; H = its central point; G = required centre of pressure. Lay off $AC = f'$ = limit of pressure. Draw triangle ACE so that its area

$= \frac{R}{b}$ = pressure per unit of breadth. Now the centre of pressure of ACE is at G, $\frac{1}{3}$ of the distance from A to E; $AG = AH - GH$
$= \frac{t}{2} - qt$ $\quad\therefore\quad AE = \frac{3}{2}t - 3qt$ and

$$ACE = \frac{f'}{2}\left(\frac{3}{2}t - 3qt\right) = \frac{R}{b}$$

$$\therefore\quad f' = \frac{2R}{\left(\frac{3}{2} - 3q\right)bt} \quad\therefore\quad q = \frac{1}{2} - \frac{2R}{3f'bt}$$

In the above case, $f' = AC > AB$ (which is double the average pressure or = what would be the max. pressure if it were zero at D). But there is another case when $f' = AC < AB$. In this case the pressure on the base is represented by ACFD a trapezoid whose average ordinate $= HK = \frac{R}{bt}$. The deviation GH in this case from eq (5) (Art. 237) is



$$GH = qt = G'H \cdot \frac{p_0 - p_2}{p_0} = G'H \cdot \frac{KH - FD}{KH}$$

where $G'H$ = deviation of the pressure was represented by ABD &

$$FD = 2HK - AC = \frac{2R}{bt} - f'$$

$$\therefore\quad qt = \frac{1}{6}t\left(\frac{bf't - R}{R}\right)$$

$$\therefore\quad q = \frac{1}{6}\left[\frac{bf't}{R} - 1\right] \quad\text{or}\quad f' = \frac{R}{bt}\left[1 + 6q\right]$$

— " —

C. E. Art. 267

From eq. (3) (Art 266. C. E), we have

$$\tan WAR = \frac{P}{W}$$ when the surface of the earth is horizontal. Also –

$$\tan W'A'R' = \frac{P}{W'}$$ for the trapezoidal wall.

But $W = whbt$ for rect. wall.

$$W' = whbt\left(\frac{1}{4} + \frac{3}{2}q\right)$$

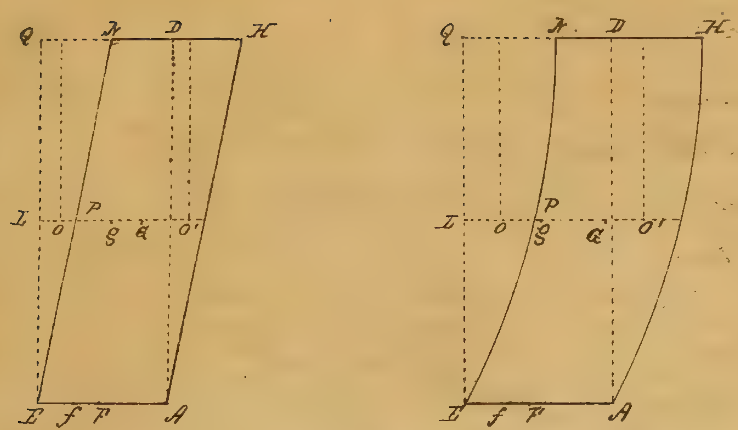
Since the mean thickness of the trapezoidal wall is

$$\frac{t + \left(3q - \frac{1}{2}\right)t}{2} = \left(\frac{1}{4} + \frac{3}{2}q\right)t$$

$$\therefore\quad \tan WAR : \tan W'A'R' :: \frac{P}{whbt} : \frac{P}{whbt\left(\frac{1}{4} + \frac{3}{2}q\right)}$$

$$:: \left(\frac{1}{4} + \frac{3}{2}q\right) : 1$$

C. E. Art. 268.



By referring to Case VIII. Art. 104, we see that the distance $Gg = OO' \frac{EQN}{EADQ}$ where OO' = the distance of the C. of G. of EQN in its first position from the C. of G. in its last position = (in this case) t.

$$\therefore \; Gg = t\,\frac{EQN}{tx} = \frac{EQN}{x}$$

— " —

C. E. Art. 269.

Volume of masonry in the counterforted wall
$= whd(bt + cT)$

Volume of masonry in the "equivalent uniform wall"
$= whd(b+c)t_1 = \sqrt{(b+c)(bt^2 + cT^2)}\,whd$

Squaring & rejecting common terms, we have
$$2Tt \gtrless T^2 + t^2$$

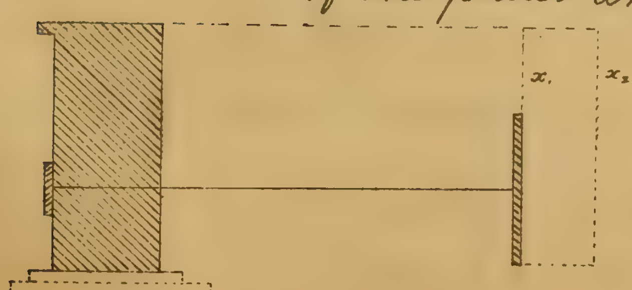
Subtracting $2Tt$ from both members, we get
$$0 < (T-t)^2$$

and since $(T-t)$ can never $= 0$, the left-hand member is always smaller than the righthand one; & therefore the quantity of masonry in the counterforted wall is always less than that in the equivalent uniform wall, & the fraction expressing their ratio is always less than unity.

— " —

C. E. Art. 271

If the plates are imbedded vertically, the pressure of the earth behind them against them will be $wx\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$. Now they can exert a pulling force on the earth in front of them = the



max. value of the ratio $\frac{p'}{wx}$ or

$$= wx\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right).$$

Of this max. pull, however, a part $= wx\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$ is = the thrust received by the plate from the earth behind & is ∴ not available for holding up the retaining wall. Therefore the net value of h = the hor. pull is

$$wx\left[\frac{1+\sin\varphi}{1-\sin\varphi} - \frac{1-\sin\varphi}{1+\sin\varphi}\right]$$

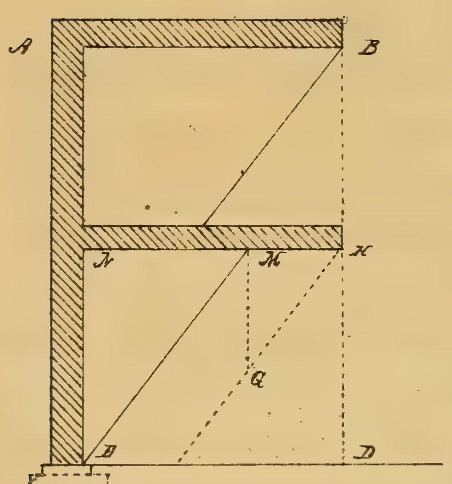
$$= wx\left[\frac{4\sin\varphi}{1-\sin^2\varphi}\right] = wx\,\frac{4\sin\varphi}{\cos^2\varphi}$$

Bet. the limits x_2 and x_1, this becomes

$$H = w.\frac{x_2^2 - x_1^2}{2}.\frac{4\sin\varphi}{\cos^2\varphi}.$$

— " —

C. E. Art. 274.



The pressure on the particle at $G = w\cos\varphi . MG$ and, as pressure is the same in every part of a layer parallel to a conj. plane, the particle at H bears upward with the same force as the one at G. (Note that when $\theta = \varphi$ in earth, the conj. pressures = each other.

∴ $MG.w\cos\varphi$ = conj. as well as vert. pressure at G)

Resolve this pressure and the hor. comp.^{c} $= MG.w.\cos^2\varphi$. This is the force that balances the hor. thrust of the earth at H; which last

$$= wx\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$$

Now calling $l = (NM + MH) = l' + l''$, we have

$$MG = l''\tan\varphi \therefore l''\tan\varphi.w.\cos^2\varphi = wx\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$$

$$\text{or } l'' = \frac{x.\cot\varphi}{\cos^2\varphi}.\frac{1-\sin\varphi}{1+\sin\varphi} = \cot\varphi.\frac{x}{(1+\sin\varphi)^2}$$

$$l' = h\cot\varphi \qquad \therefore l = \cot\varphi\left(h + \frac{x}{(1+\sin\varphi)^2}\right).$$

C. E. Art. 281.

When the form of a linear rib suited to sustain a given load is assumed for the intrados of a real arch, it is necessary to determine whether the actual line of pressures will lie within the middle third of the arch ring.

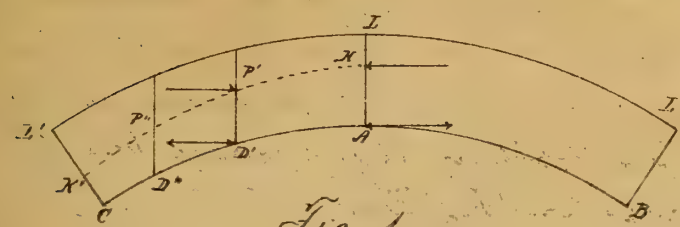


Fig. 1

To ascertain this in the case of the parabola & catenaries, we consider any part of the arch included bet. the vert. plane AL at the crown & a vert. plane at any other point as D'P'. The calculated hor. thrust along the linear rib which coincides in shape with the soffit CA is indicated by the arrow with head at A. Let the hor. thrust of the rib at D' be indicated by the arrow there pointing in an opposite direction to that at A. Take $AK \geq \frac{2}{3} AL$ & imagine a left-handed couple, applied to AL in the vert. plane of the arch, whose force = H = thrust at A & whose lever-arm = AK. Applying equal & opposite couples on the plane D'P' with a force H' = hor. thrust of the rib at D' its lever-arm D'P' must then $= H \cdot \frac{AK}{H'}$. In the parabola & catenaries $H = H' \therefore D'P' = AK$. These couples, being equal & opposite, do not change the conditions of equilibrium of the section of the arch LD', but they transfer the line in which the thrust acts from AD' to KP'. We can repeat the process as often as we choose by taking parts LD'' &c, &, if the curve drawn thro. the points K, P', P'' &c lies within the middle third, the arch will be sufficiently stable. In the case before us, since H is constant, the lever-arms D'P' &c are constant too & the line of pressures KK' will be merely the parabola or catenary raised a vert. distance = AK. If KK' does not lie in

the middle third, a slight increase in the depth of the voussoirs, especially towards the springing, will usually remove the difficulty. In the case of the circle & hydrostatic curve take the plane D'P' normal to the curve & apply forces of the second couple at D' & P' in the direction of the tang. to the intrados at D'. Then, since the thrust around the circular or hydrostatic rib is constant, we have the lever-arm AK = D'P' or the line of pressures will be parallel to the intrados & of course will lie within the middle third. The general case is discussed in the text.

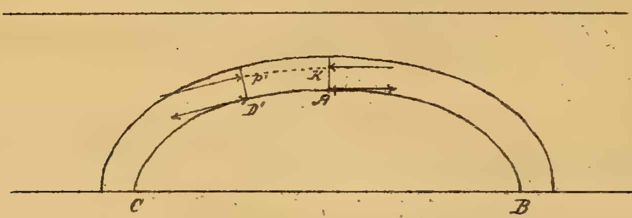


Fig 2

C. E. Art. 282.

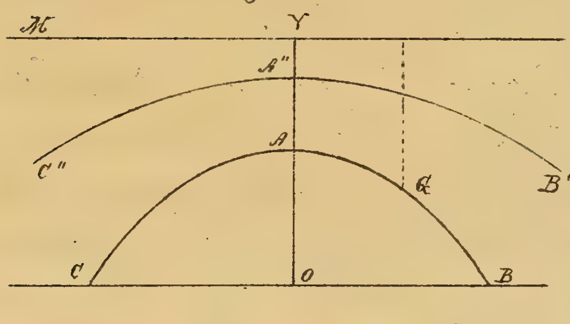


Fig. 3

The extrados of the Transformed Catenary need not be the directrix MY. It may be another transformed cat.^{y} provided these catenaries have the same directrix. To illustrate — Suppose the wt. of a unit of the material bet. CAB & MY = w. Then the intensity of vert. pressure at any point a of CAB = wy. If a ~~one~~ heavier building material were used, this vert. pressure could be brought upon a by a less height of it. Let a unit's wt. of the heavier material = w', & let $w' = \frac{3}{2} w$; then a column of the heavier material whose height = $\frac{2}{3} y$ would give the same pressure as the entire column of the lighter. or

$$wy = \tfrac{2}{3} w' y$$

At each point of CAB lay off 2/3 of the vert. ordinate & thro. these points draw C"A"B". The upper surface of the load may have this form & CAB still be balanced under the forces, since the eq. of CAB is

$y = \frac{y_0}{2}(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}})$. That of $C''A''B''$ is evidently

$y' = \frac{3y_0}{2}(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}})$. The principle of this example is general.

When the extrados is the transformed cat? note that, since in all the formulae of Art. 131 w = wt. corresponding to a unit of surface of the space bet. CAB & MY, we must make in those formulae

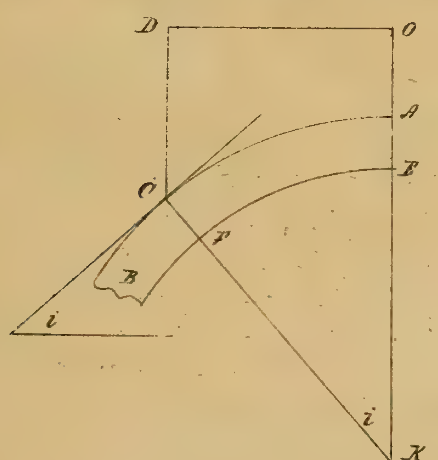
$$w = nw'$$

where w' is the wt. of a unit of building material & $n = \frac{AA''}{YA}$.

— " —

C. E. Art. 286.

If we discuss the arch as we did in Sc. IV. Art 138. we have:— the vert. pressure on arch bet. the crown & any point C due to the loading alone $= w'\int(c+x)\,dy$.



That due to the wt. of the archstones from A to C = w(area $AECF$) $= \frac{wi}{2}(r'^2 - r^2)$

Hence total vert. pressure

$P_x = w'\int(c+x)dy + \frac{wi}{2}(r'^2 - r^2)$

Substitute for x & dy their values in terms of i &

$$P = w'r'(c\sin i + r'(\sin i - \cos i\frac{i\sin i}{2} - \frac{i}{2})) + \frac{w}{2}i(r'^2 - r^2)$$

and

$$P\cot i = w'r'\left[c.\cos i + r'(\cos i - \frac{\cos^2 i}{2} - \frac{i}{2}.\frac{\cos i}{\sin i})\right] + \frac{w}{2}.\frac{i\cos i}{\sin i}(r'^2 - r^2)$$

Hence $p_y = \frac{dP_y}{dx} = -\frac{\delta(P\cot i)}{dx} = -\frac{1}{r'\sin i}.\frac{d(P\cot i)}{di}$

$$= -\frac{1}{r'\sin i}.\frac{d}{di}\left[w'r'\left[c.\cos i + r'(\cos i - \frac{\cos^2 i}{2} - \frac{i\cos i}{2\sin i})\right] + w\frac{i\cos i}{2\sin i}(r'^2 - r^2)\right]$$

$$= w'c - w'r'\left[-1 + \cos i + \frac{i - \cos i\sin i}{2\sin^3 i}\right] + \frac{w}{r'}\left[r'^2 - r^2\right]\left[\frac{i - \cos i\sin i}{2\sin^3 i}\right]$$

Place this $= 0$ & multiply by $\frac{r'}{w}$, then

$$p_y = 0 = \frac{w'}{w}cr' - \frac{w'r'^2}{w}\left[-1 + \cos i_0 + \frac{i_0 - \cos i_0\sin i_0}{2\sin^3 i_0}\right] + (r'^2 - r^2)\left(\frac{i_0 - \cos i_0\sin i_0}{2\sin^3 i_0}\right).$$

$$= r'^2\left[\frac{w'}{w}(1-\cos i_0) + \left(1-\frac{w'}{w}\right)\left(\frac{i_0 - \cos i_0 \sin i_0}{2\sin^3 i_0}\right)\right] + \frac{w'}{w} c r' - \left[\frac{i_0-\cos i_0 \sin i_0}{2\sin^3 i_0}\right] r^2 = 0$$

If T_0 = thrust at crown, then

$$P_y = T_0 - P\cot i$$

But $T_0 = w'cr' + [r'-r]wr$: hence

$$P_y = w'cr' + wr(r'-r) - w'r'\left(c.\cos i - r'\left(\cos i - \frac{\cos^2 i}{2} - \frac{i\cot i}{2}\right)\right) - \frac{w}{2}\, i\cot i\,(r'^2-r^2)$$

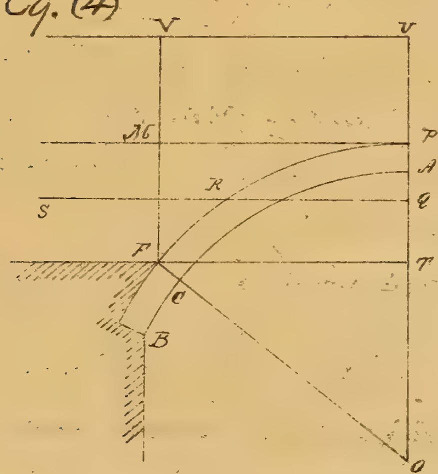
Therefore

$$H_0 = T\cos i = T_0 - P_y = w'cr' + wr(r'-r) - P_y$$

$$= w'r'\left[c.\cos i_0 + r'\left(\cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0\cot i_0}{2}\right)\right] + \frac{w}{2}\, i_0 \cot i_0\,(r'^2-r^2)$$

$$= w'r'^2\left[\left(1+\frac{c}{r'}\right)\cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0 \cot i_0}{2}\right] + w(r'^2-r^2)\frac{i_0\cot i_0}{2}.$$

Eq. (4)



The spandrie FVUP is composed of two parts —

1°– The rect. MVUP.

2° The surface FMP.

$FT = OT = r'\sqrt{1/2}$. $PT = PO - TO = r'(1-\sqrt{1/2})$

Hence area of sector OFP

$$= \frac{1}{8}\cdot 2\pi r'\cdot\frac{r'}{2} = .3926\, r'^2$$

Area of triangle OFT

$$= \frac{(r'\sqrt{1/2})^2}{2} = .25\, r'^2$$

∴ Segment FTP $= .1426\, r'^2$

Area of rect. FMPT $= PT.\,FT = .2071\, r'^2$

∴ area FMP $= .0645\, r'^2$

Again $MP = FT = r'\sqrt{1/2}$

∴ area MVUP $= c r'\sqrt{1/2} = .7071\, c r'$

Also

area of ring FCBA $= \frac{1}{8}(\pi r'^2 - \pi r^2) = .3927(r'^2 - r^2)$

Hence &c

C.E. Art. 339.

General Case

Fig. 1

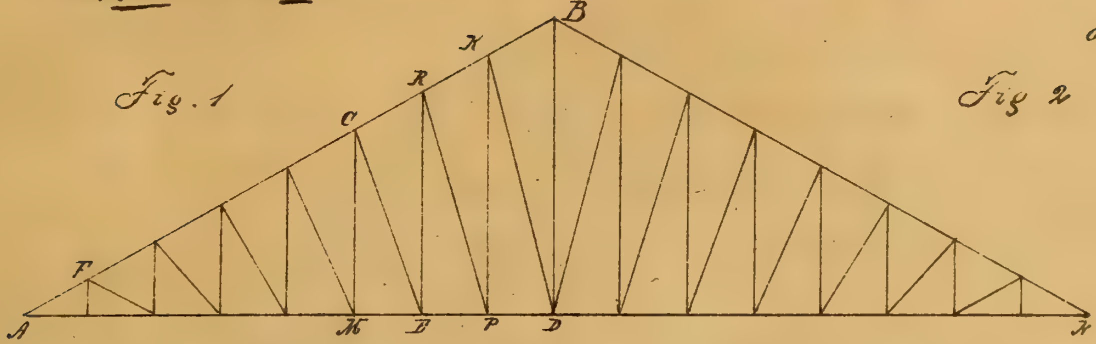
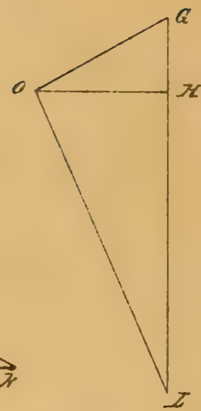


Fig 2



The wt. on any section of the rafter like AF or CR $= \frac{W}{2n}$. The wt. supported at any point C by the subordinate truss whose vertex is at that point is $=$ $\frac{1}{2}$ the wt. on AC plus ½ the wt. on CR. The number of sections in AC $= (n-m)$. Add 1 & the wt at C $=$

$$\frac{n-m+1}{2} \cdot \frac{W}{2n} = \frac{n-m+1}{4n} \cdot W$$

At K (since m is there $= 1$) this becomes

$$\frac{n}{4n} \cdot W = \frac{W}{4}.$$

Of this wt. the part transmitted to D is $\frac{W}{4} \cdot \frac{n-1}{n}$ & since an equal amt. comes to D from the corresponding truss on the right side of the roof, we have vert. pull on BD

$$= 2\frac{W}{4} \cdot \frac{n-1}{n} = \frac{W}{2}\left(1 - \frac{1}{n}\right).$$

Add this to $\frac{W}{2n}$ for total load at B.
GOL is the triangle of forces for the truss ACE. In this the small triangle OGH is similar to ACM & so is OHL to CME. Hence

$$CM : GH :: AM : OH$$

$$\text{and} \quad CM : ME :: HL : OH$$

$$\text{whence} \quad \overline{GH} \cdot \overline{AM} = \overline{ME} \cdot \overline{HL}.$$

$$\text{and} \quad GH : HL :: ME : AM :: CR : AC :: 1 : n-m.$$

$$\therefore \quad GH : GH + HL :: 1 : n-m+1$$

$$\therefore \quad GH = \frac{1}{n-m+1} \cdot GL = \frac{1}{n-m+1} \cdot \frac{n-m+1}{4n} \cdot W = \frac{W}{4n}.$$

Since $GL = $ wt. at C $= \frac{n-m+1}{4n} \cdot W$).

$$OH = H_m = \frac{W}{4n} \cdot \frac{c}{k} \quad \text{which is constant.}$$

$$T_m = OG = \sqrt{\overline{GH}^2 + \overline{OH}^2}.$$

$$HL = GL - GH = \frac{n-m}{n-m+1} \cdot GL$$

$$= \frac{n-m}{n-m+1} \cdot \frac{n-m+1}{4n} W = \frac{n-m}{4n} W.$$

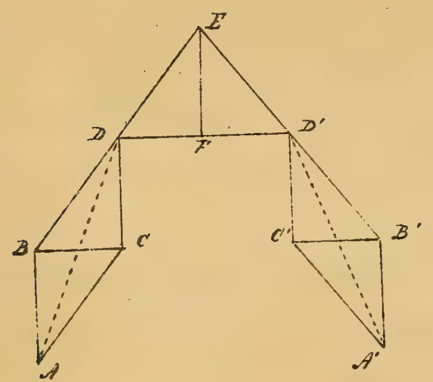
$$OL = S_m = \sqrt{\overline{OH}^2 + \overline{HL}^2} = \sqrt{H_m^2 + \frac{W^2}{16n^2} \cdot (n-m)^2}$$

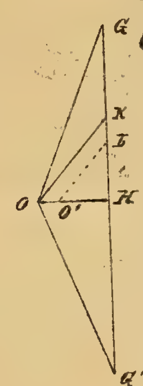
The total tens.ⁿ at P is that due to the primary system ABN & that due to the secondary system AKD $= H + H_m$
$= H(1 + \frac{1}{n})$.

Bet D & P the system ARP also acts. Hence the hor. stress $= H + 2H_m = H(1 + \frac{2}{n})$ & so on. The general term is
$$H\left(1 + \frac{n-1}{n}\right) = H\left(\frac{2n-1}{n}\right).$$

Gothic Roof Truss.

The primary truss ADD'A' bears the wts at D, E & D'. One half the whole load rests on the rafter BE, & ∴ 1/4 on BD. Hence at E the load supported is $= 2/8 = 1/4$. At D there is also $2/8 = 1/4$ & the same at D'. Hence the total load upon the primary truss $=$ 6/8 of W.



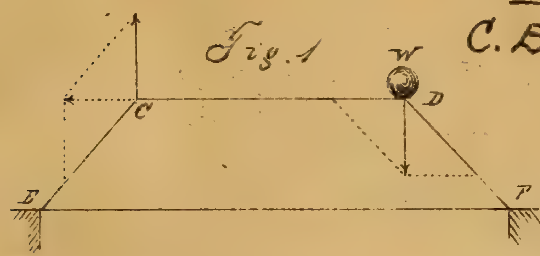


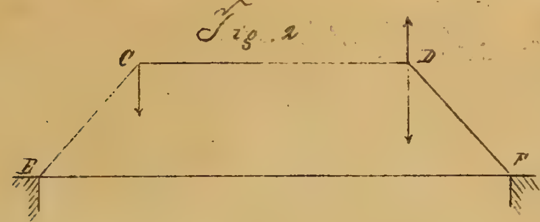
Draw triangle of forces GOG' making GG' = 6/8 of W. Then OH = thrust on DD' & OG = thrust in direction AD.

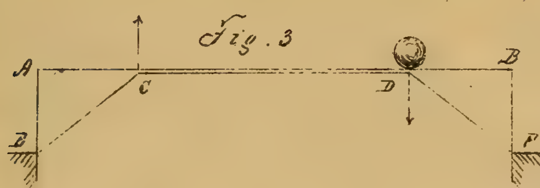
Draw OK parallel to BD (K bisects HG since DC = AB). It represents the thrust on DB & OKH is the triangle of forces for the pieces BD, DC & BC. The thrust on BA = that on DC + 1/8 W originally supported at B.

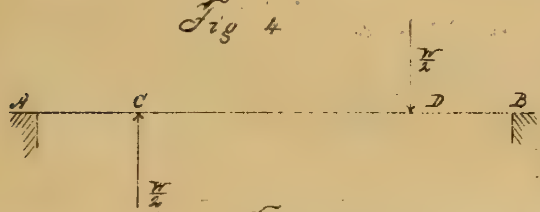
The load on the secondary truss DED' $= 1/4$ W. Then 1/3 of HG = 1/2 of this & 1/3 HG = 2/3 HK &, as DE is parallel to OH, the hor. tens.ⁿ in the triangle of forces this truss = O'H = 2/3 OH ; ∴ the resultant thrust on

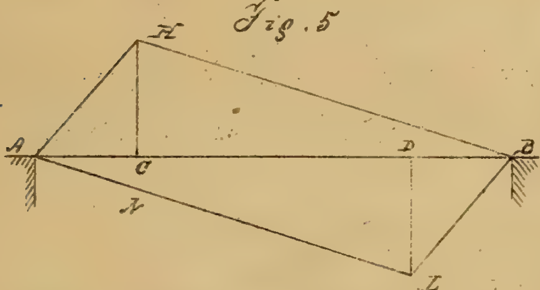
$DD' = OH - \frac{2}{3}OH = \frac{1}{3}OH$, & that along $DE = \frac{2}{3}OH$











" C.E. Art. 341.

Load unequal longitudinally

A wt. W at D (Fig. 1) on a frame hinged at C & D produces a downward thrust at D & an equal upward & opposite one at C. Now this downward push at D can be neutralized :- either by an equal & opposite pull at D or by a downward push at $C = \frac{W}{2}$ + a pull at $D = \frac{W}{2}$ (Fig. 2.) For the downward force at C ($= \frac{W}{2}$) produces an upward one at D ($= \frac{W}{2}$) & this together with the direct upward force there will be equal & opposite to the entire wt. (W). The latter is the way in which the beam AB (Fig. 3) (or a tie beam when the truss is constructed with one) neutralizes the action of the wt. viz-
By exerting equal resistances to the upward push at C & the downward pull at D.

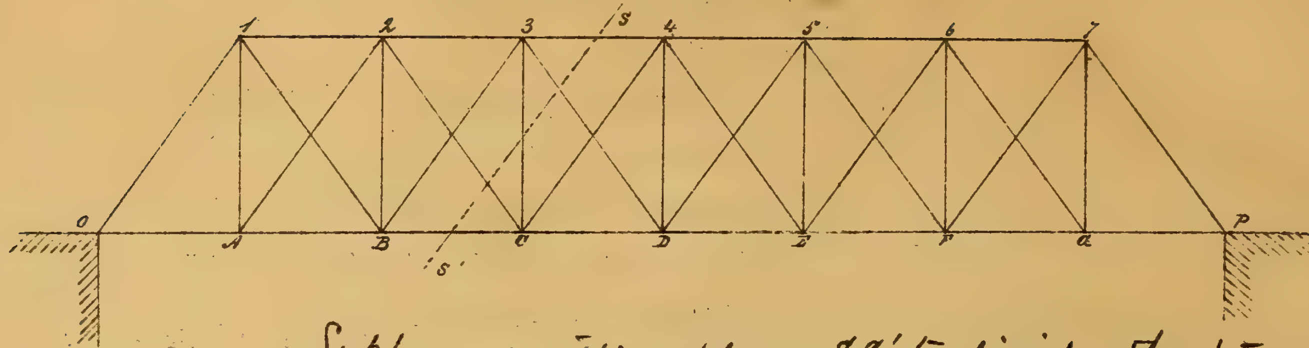
Hence AB (Fig. 4) is acted on by two opposite forces each $= \frac{W}{2}$, the one at D & the other at C.

The mom^ts due to these two forces are shown in Fig. 5., where HC = mom^t due to upward force at C & DI = HC = that due to downward force at D.

The total mom^t at C or D, where it is evidently a max., is

$$\frac{W}{2}\cdot\frac{c+x}{2c}(c-x) - \frac{W}{2}\cdot\frac{c-x}{2c}(c-x) = \frac{Wx}{2c}(c-x) \quad \text{.........} (3)$$

C. E. Art. 343.



Suppose a cutting plane SS' to divide the truss as in the fig. intersecting the top chord in the n^{th} panel. The max. momt. in every part of the bridge occurs when the whole load is on, & in that case the counterbraces do not act. Hence. the plane SS' cuts but three acting pieces: viz - the upper & lower chords & the n^{th} vertical; & the stresses in these pieces together with the vert. forces on either end of the truss (the left for instance) must satisfy the conditions of equilibrium. These are

$$\Sigma X = 0 \quad , \quad \Sigma Y = 0 \quad , \quad \Sigma M = 0$$

Take C as the centre of mom.ᵗ The external forces are - the reaction of the abutment & the load bet. C & the end. at O. The reaction is $(w+w')\frac{N-1}{2}$. Its lever-arm about $C = n.\frac{l}{N}$.

The load bet O & C (omitting the wt. at C, since its leverage = 0) is = $(w+w')(n-1)$ (= wts at A & B). The lever-arm of this load = $\frac{n}{2}.\frac{l}{N}$.

The stresses in the n^{th} vert. and the lower chord, both passing thro. C, have no mom.ᵗˢ about that point. The upper chord stress = H_n. Its lever-arm = k

$$\therefore \Sigma M = (w+w').\frac{N-1}{2}.n.\frac{l}{N} - (w+w')(n-1).\frac{n}{2}.\frac{l}{N} - H_n k = 0$$

$$\therefore H_n = \frac{(w+w')l}{k}.\frac{n(N-n)}{2N} \quad \text{-----------} (1)$$

It is apparent that the same stress exists in bay BC of the lower chord. When the load is on the top of the truss, we have precisely the same eq. & hence the same value for H_n.

Shearing stress for permanent load, $\Sigma Y = 0$.

When the load is on the bottom chord –

$$w\left(\frac{N-1}{2}\right)-(n-1)w-V_n'=0=\Sigma Y$$

$$\therefore V_n'=w\left(\frac{N+1}{2}-n\right)$$

When the load is on the top –

$$V_n'=w\left(\frac{N-1}{2}\right)-nw=w\left(\frac{N-1}{2}-n\right)$$

For passing load,

The max. shearing force due to the passing load at the section SS′ occurs when the right-hand end of the truss only is loaded. In this case, too, there is no stress on the counterbraces cut by SS′. Hence the only vert. forces in the left-hand end of the truss are the reaction of the abutment, due to the partial passing load, & the stress in the n^{th} vert. When the load is below, the load from SS′ to the right-hand end $= w'(N-n)$. The C. of G. of this load is distant $\frac{N-n+1}{2}$ panels from P. Hence the reaction of the left abutment is

$$= w'(N-n)\frac{N-n+1}{2}\cdot\frac{l}{Nl} = w'(N-n)\frac{N-n+1}{2N}$$

$$\therefore V_n''=w'(N-n)\frac{N-n+1}{2N}.$$

When the load is above, the passing load on the right-hand end is $= w'(N-n-1)$. Its C. of G. is distant from P $\frac{N-n}{2}\cdot\frac{l}{N}$. Hence the reaction at O due to it

$$= w'(N-n-1)\frac{N-n}{2}\cdot\frac{l}{Nl} = w'(N-n)\frac{N-n-1}{2N}$$

$$\therefore V_n''=w'(N-n)\frac{N-n-1}{2N}.$$

Hence for load below –

$$V_n=V_n'+V_n''=w\left[\frac{N+1}{2}-n\right]+w'(N-n)\frac{N-n+1}{2N} \quad\text{--------}(2)$$

For load on top –

$$V_n=V_n'+V_n''=w\left[\frac{N-1}{2}-n\right]+w'(N-n)\frac{N-n-1}{2N}$$

The difference of these is $= w+w'\left(\frac{N-n}{N}\right)$

At the middle $n = \frac{N}{2}$. For load below –

$$V_n = \frac{w}{2} + w'.\frac{N}{2}\left(\frac{\frac{N}{2}+1}{2N}\right) = \frac{w}{2} + w'\left(\frac{N}{8}+\frac{1}{4}\right) \text{ - - - - - - - - - - - (3)}$$

$$\left.\begin{array}{l}\text{When load is on lower chord , } T_n = \frac{s}{k}\cdot V_n \\ \text{" " " " top " , } T_n = \frac{s}{k}\ V_{n-1}\end{array}\right\} \text{ - - - - - - - (4)}$$

These values are identical.

The n^{th} counterbrace bearing the stress (t_n) is $3\bar{D}$ & the brace in the same panel is $4\bar{o}$ which is the $(n+1)^{th}$. The only load tending to strain $3\bar{D}$ is the passing load on A_1, B, C when these vertices are alone loaded; & even then no stress exists in the counterbrace unless the action of this partial passing load exceed that of the permanent load in the $(n+1)^{th}$ brace. The effect of the permanent load on the $(n+1)^{th}$ brace is

$$T'_{(n+1)} = \frac{s}{k} V'_{(n+1)} = w\frac{s}{k}\left(\frac{N-1}{2} - n\right)$$

The passing load acting on $3\bar{D} = nw'$. Its C. of G. is $\frac{n+1}{2}$ panels from O. Hence the reaction of P due to this load is

$$nw'.\frac{n+1}{2}.\frac{l}{Nl} = w'\frac{n(n+1)}{2N}$$

& the force in the direction of the counterbrace is

$$\frac{s}{k}.w'.\frac{n(n+1)}{2N}.$$

Hence

$$t_n = \frac{s}{k}.w'.\frac{n(n+1)}{2N} - \frac{ws}{k}\left(\frac{N-1}{2} - n\right) \text{ - - - - - - - - - - - - - - (5)}$$

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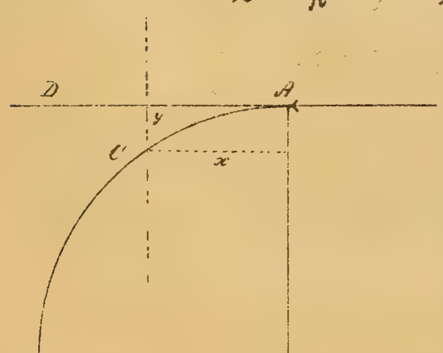


C. E. Art. 345.

The mom.^{t} of the external forces at any point C of the quadrant, if both ends are hinged, is (taking mom.^{ts} on the right of a vert. plane thro C.) found thus:–

1° The hor. thrust at the crown A of which the mom.^{t} is Hy (y being the ordinate of C from AD). 2° The uniform load along $AD = wx$. of which the mom.^{t} is $= \frac{wx^2}{2}$

These act in opposite directions & their difference is

$$\frac{wx^2}{2} - Hy = M \text{ (the mom}^t \text{ of resistance of rib)}$$

Now

$$H = \frac{wr}{2} \quad \text{and} \quad x^2 = 2ry - y^2$$

$$\therefore \quad \frac{w}{2}(2ry - y^2) - \frac{wr}{2}\cdot y = M.$$

To find the max. value of this -

$$\frac{dM}{dy} = -2y + r = 0 \quad \therefore \; y = \frac{r}{2}$$

Hence - Max. mom^t = $M_0 = \frac{3}{8}wr^2 - \frac{1}{4}wr^2 = \frac{1}{8}wr^2$ - - - - (2)

— " —

C. E. Art. 346

The method of the text for the stresses in the diag.^{s} seems but a coarse approximation. The arch when a parabola being equilibrated under uniformly distributed load, these pieces are called into action only by the travelling load as it passes over the bridge.

To find the comp.^{n} in any diag. strut, sloping like HE, in the segment BC.

Draw the tang. FO at F & imagine a cutting plane QQ' just to the left of EF. The load, producing max. comp.^{n} in HE, extends from A up to & including D but not H. The only force to the right of QQ' is the reaction R of the abutment at C, & this with the stresses in the three beams cut by QQ' must be balanced. Let the letters have the same signification as in the text.

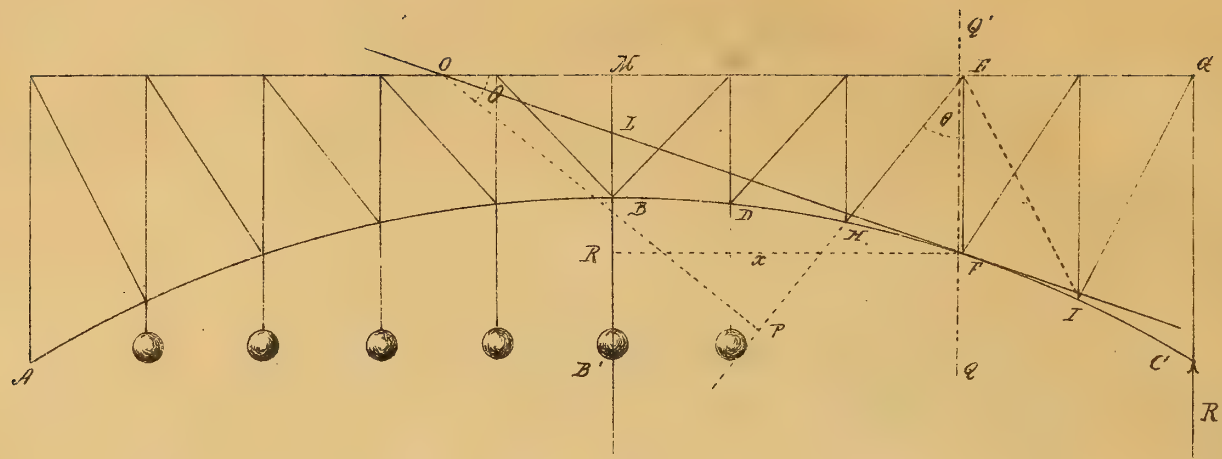
Then

$$R = (n-1)w' \cdot \frac{n}{2N}.$$

Take B, the vertex of the parabola, as the origin & let the distance BM = a. $x^2 = 2py$ is the eq. of the arch; & if k_1 = the rise BB' & l = the span AC, this eq. becomes

$$x^2 = \frac{l^2}{4k_1}y \quad \text{or} \quad y = \frac{4k_1}{l^2}\cdot x^2.$$

Counting x (the hor. distance of the plane QQ'



from B) as = RF, we have RL = 2 BR = 2y (since the subtang. is bisected at the vertex), & the triangles OEF & FLR being similar,

$$2y : x :: EF : EO :: a+y : EO = \frac{x(a+y)}{2y}$$

Again $GO = EO + \left(\frac{l}{2} - x\right) = \frac{x(a+y)}{2y} + \left(\frac{l}{2} - x\right) = \frac{ly + ax - xy}{2y}$.

and OP = lever-arm of stress in HE about O

$$= EO \cos\theta = \frac{x(a+y)}{2y} \cdot \cos\theta$$

Hence mom^{ts} around O (if T = stress in HE)

$$T\left(\frac{x(a+y)}{2y}\right)\cos\theta = R\left(\frac{ly + ax - xy}{2y}\right)$$

$$T = R \cdot \frac{ly + ax - xy}{ax + xy} \cdot \sec\theta = \frac{n(n-1)}{2N} \cdot w' \cdot \frac{ly + ax - xy}{ax + xy} \cdot \sec\theta$$

The values of x, y, & $\sec\theta$ are obtained from the eq. of the parabola or from a carefully constructed figure.

A similar analysis would give the stresses on the counterbraces (as EI when the shorter segment of the bridge is loaded); but these may without serious error be made of the same size as the braces.

The max. compression on a post as EF
$= w'' + w'$.

The max. tension, if any, on EF
$= T \cos\theta - w''$.

C. E. Art. 347.

See discussion of Bow-string Girder under iron bridges (Art. 379)

C.E. Art. 348

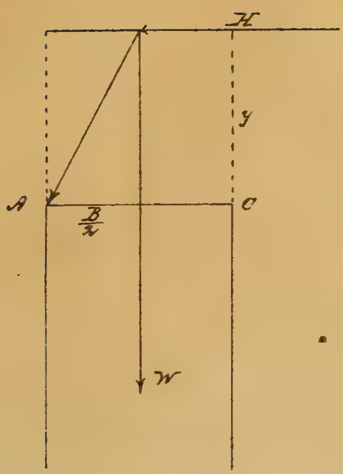


Fig. 1

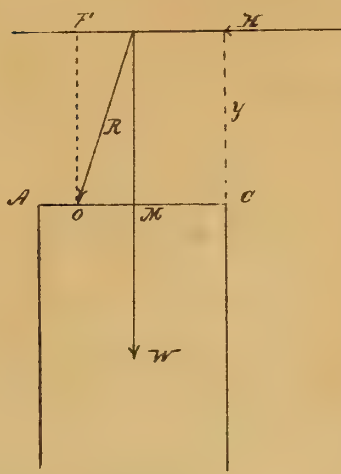


Fig. 2

Fig. 1. represents the case of eq. 1. Here the resultant of H & W passes thro. A, & if we resolve R into its vert. & hor. comp.^s at A, the vert. = W = wt. on post at A. There is no stress at C. Taking mom.^s about A

$$Hy = W \cdot \frac{B}{2}.$$

$$\therefore B = \frac{2Hy}{W}$$

Fig. 2. represents the case of eq. 2, the resultant of H & W passing thro. O whose distance from M place = u. Decompose R at O into OF (= W) & OM. Take mom.^s around C. Then calling the reaction at A = P, we have

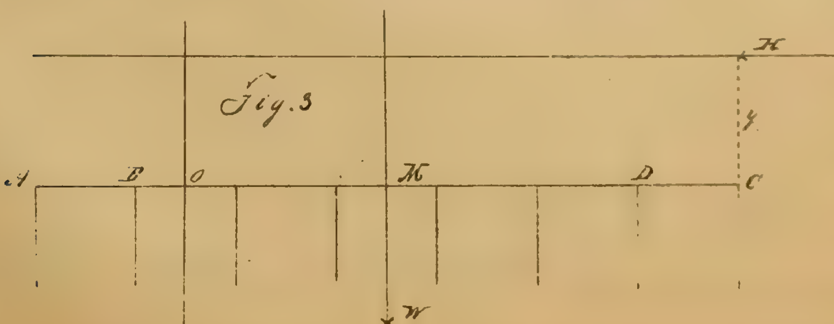
$$P.B = W.\overline{OC} = W\left[\frac{B}{2}+u\right]$$

But $\quad Wu = Hy$

$$\therefore P.B = W.\frac{B}{2} + Hy$$

$$\therefore P = \frac{W}{2} + \frac{Hy}{B}.$$

Eq. 3. As in the last case, the resultant of H & W is equivalent to a vert. force = W at O & a hor. force which, of course, does not affect the downward pressure upon the piers. As in Art. 341, denote the distances from the centre M to each row of posts going towards A by $u_1, u_2 \ldots u_n$ respectively; & similarly let the distances towards C be $-u_1, -u_2, -u_3 \ldots -u_n$. Let $MO = u_0$.



Then in accordance with the discussion of a uniformly varied stress (A.M. Art. 94) the deviation of the load on any row of

posts from the mean load $\left(\frac{W}{n}\right)$ may be called $a x$ & the constant a can be found as follows; –

$$Wx_0 = ax_1^2 + ax_2^2 + \cdots\cdots ax_n^2 + ax_1^2 + ax_2^2 \cdots\cdots ax_n^2$$
$$= 2a\,(x_1^2 + x_2^2 + \cdots\cdots x_n^2)$$

and since $Wx_0 = Hy$

$$a = \frac{Hy}{2(x_1^2 + x_2^2 + \cdots\cdots x_n^2)}$$

The load on the post at A

$$= \frac{W}{n} + ax_n = \frac{W}{n} + \frac{Hyx_n}{2(x_1^2 + x_2^2 + \&c)} \cdots\cdots\cdots\cdots (1)$$

But $x_n = MA = \frac{B}{2}$. Also

$$x_{n-1} = MA - AE = \frac{AC}{2} - \frac{(AE + DC)}{2} \quad \& \left(\text{since } AE = \frac{B}{n-1}\right)$$
$$= \frac{B - \frac{2B}{n-1}}{2} = \frac{B(n-3)}{2(n-1)}$$

Similarly

$$x_{n-2} = \frac{B(n-5)}{2(n-1)} \quad \&c$$

Substitute in (1) and

$$\frac{W}{n} + ax_n = \frac{W}{n} + \frac{Hy.\frac{B}{2}}{2\left[\left(\frac{B}{2}\right)^2 + \frac{B^2(n-3)^2}{4(n-1)^2} + \frac{B^2(n-5)^2}{4(n-1)^2} + \&c\right]}$$
$$= \frac{W}{n} + \frac{Hy(n-1)^2}{B\left[(n-1)^2 + (n-3)^2 + \&c\right]} = P$$

and
$$P' = \frac{W}{n} - ax_n = \frac{W}{n} - \frac{Hy(n-1)^2}{B\left[(n-1)^2 + (n-3)^2 + \&c\right]}$$

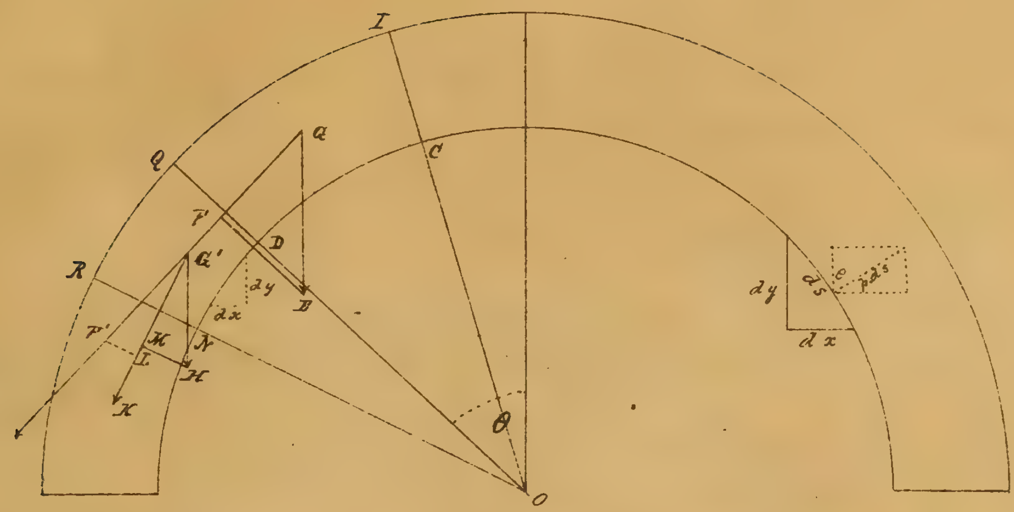
— " —

C. E. Art. 349

Let RD be a stone of the arch whose length on the intrados = 1 ft. Then its wt. = w. Lay off C'H vertically to represent this wt. & resolve it in the direction of the normal & tang. to the arch at N. This is done in the triangle C'NH. The normal compt is

$$NH = w.\cos\theta.$$

This will be the pressure on the centre for the distance



ND if the arch has been built no higher than D. But if the arch has been carried up to C, the pressure of the part CD tends to push out the stone RD & in so far, counteracts the above normal thrust. Thus, let $GE = W$ = total wt. of the part $CDQI$. If the arc $= s$, then $W = \int w ds$.

Resolve this wt. in the direction of the normal & tang. at D. This is done in the triangle GFE; FG, the tangential compt., is the pressure brought by the upper part of the arch ring on the joint QD.

$$FG = W \sin\theta = \int w ds . \sin\theta .$$

By reference to the differential triangle at D, we see that

$$ds . \sin\theta = dy$$

$$\therefore \quad FG = \int_{y_0}^{y} w dy .$$

Prolong GF & from G' where it intersects $G'H$, lay off $G'F'$ on it $= GF$. Draw $G'K$ parallel to the tang. at N & resolve $G'F'$ parallel & perp. to this line. Then $F'L$ is evidently the compt. of $G'F'$ which tends to make the stone RD slide up the joint at R, &, in so far, neutralizes the normal compt. of the wt. of that stone. This normal compt. was found above to be $MH = w\cos\theta$.

Hence the net normal pressure along the arc ND (1 ft. in length) is $p = w\cos\theta - F'L$.

But $F'L = F'G' \sin F'G'L$, &, since KG' & $F'G'$ are perp. to QO & RO, $F'L = F'G' \sin ROQ$. We may take $\sin ROQ = \frac{ND}{r} = \frac{1}{r}$ since the angle is small.

Therefore $$F'L = F'C' \cdot \frac{1}{r} = \frac{1}{r}\int_{y_0}^{y} w\,dy$$

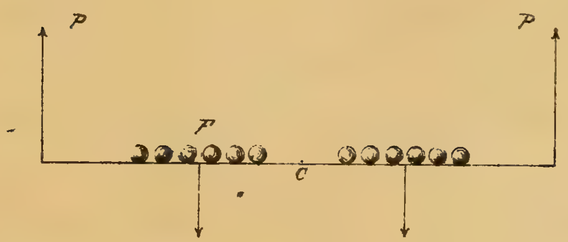
$$\therefore\ p = w\cos\theta - \frac{1}{r}\int_{y_0}^{y} w\,dy$$

The max. value of this is, of course, $p = w\cos\theta$

Eq (2) (Art. 349) The normal pressure on an arc $ds = p\,ds$. The vert. compt. of this $= p\,ds\cdot\cos\theta$. But $\cos\theta.ds = dx$ $\therefore$ vert. compt. $= p\,dx$

$$\therefore\ P = \int_{x_0}^{x} p\,dx$$

Eq. 3.



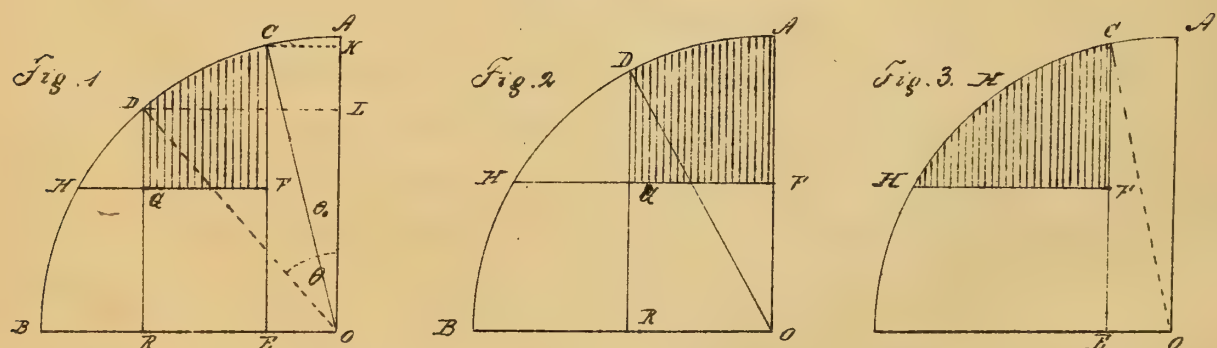
The reaction of each abutment $= P$. The wt. at F distributed over a portion of the rib has an intensity $= p\,dx$. Hence, taking momt. around C, we have

$$M = Pc - \int_{x_0}^{x} c.p.dx$$

and when the load extends to the crown $$M_1 = P_1 c - \int_0^{x} p.x.dx \; .$$

Eq.s (5) (6) (7) are integrated by means of the formula for trignometrical functions.

Geometrical Illustrations—



In Fig. 1. make $AO = 2w$. Then $CE = 2w\cos\theta_0$, &, drawing FH thro. the middle point of CE, we have

$$CF = FE = w\cos\theta_0\,.$$

Again $\quad DR = 2w\cos\theta$ (DOA being $= \theta$).

$$\therefore\ DG = 2w\cos\theta - w\cos\theta_0 = p \text{ - - - - - - - - - (5)}$$

This expression for p is general, and

$\therefore \quad P = \int_{x_0}^{x} p\,dx = \text{area } CFGD.$

The shaded surface CHF (Fig. 3) gives the entire wt. down to the point at which the arch-ring ceases to press on the centre; for at H $\quad 2w\cos\theta - w\cos\theta_0 = p = 0$

When the arch ring is built up to the key-stone at A (as in Fig. 2), bisect AO & draw FH horizontal. Then, since $\theta_0 = 0$, we have

$$p = DG = 2w\cos\theta - w \quad ,$$

and the load on the centre from A to D is

$$P_1 = \int_0^{x_1} p\,dx = \text{area } AFGD.$$

The entire pressure on the centre from A to H, the point of no pressure, is

The area AFH.

The value of $HOA (= \theta_1)$ (Fig. 2.), since $FO = \frac{AO}{2}$, is $= 60°$

Hence no provision need be made for pressure below the point whose inclination is $60°$, when the arch ring is complete.

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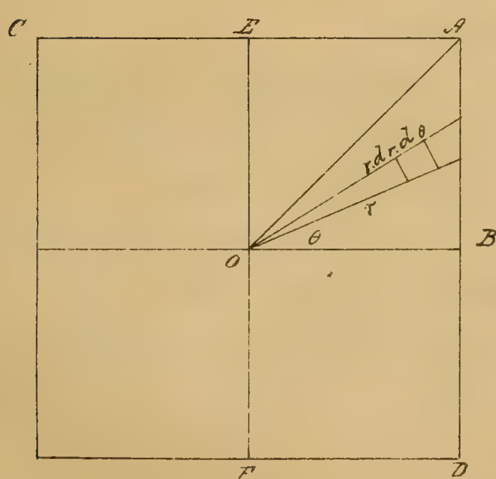
C. E. Art. 349 A



The mom.t of resistance of the triangle OAB = 1/8 of that of the square CD. Let $b = OB = \frac{h}{2}$. Also f = stress at A. Then Stress on element $r\,dr\,d\theta$ = $\frac{fr}{b\sqrt{2}}$. Hence

$$\frac{M}{8} = \iint r\,dr\,d\theta \cdot \frac{fr}{b\sqrt{2}} \cdot r.$$

$$= \int_{r=0}^{r=b\sec\theta}\int_{\theta=0}^{\theta=45°}\left(\frac{r^3 f}{b\sqrt{2}}\,dr\,d\theta\right)$$

$$= \frac{f}{b\sqrt{2}}\int \frac{r^4}{4}\,d\theta = \frac{f}{b\sqrt{2}}\int \frac{b^4\sec^4\theta}{4}\,d\theta$$

$$= \frac{fb^3}{4\sqrt{2}}\int \sec^4\theta\,d\theta = \frac{fb^3}{4\sqrt{2}}\left[\frac{\sin\theta}{3}\left(\frac{1}{\cos^3\theta} + \frac{2}{\cos\theta}\right)\right] = \frac{fb^3}{4\sqrt{2}}\cdot\frac{4}{3} = .236\,fb^3$$

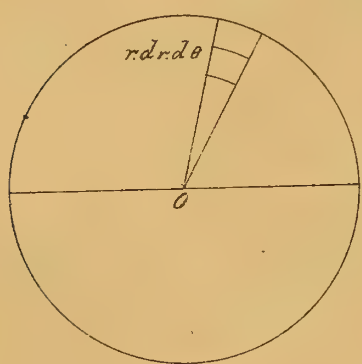
$$\therefore \quad M = 8\left[.236\,f\frac{h^3}{8}\right] = .236\,fh^3$$

Eq. (2) $\qquad \frac{\theta}{l}$ = angle of twist bet. two sections whose

distance apart equals unity. $\frac{\theta}{l}\cdot h\frac{\sqrt{2}}{2}$ = space described by the fibre most twisted (that at A) = "strain". Then $\frac{\theta}{l}\cdot\frac{h\sqrt{2}}{2}\cdot C$ = force producing this strain.

$$\therefore \frac{\theta}{l}\cdot\frac{h\sqrt{2}}{2}\cdot C = f' = \frac{M'}{.236\,h^3} \quad \therefore \theta = \frac{M'l}{Ch^4}\cdot\frac{\sqrt{2}}{.236} = \frac{M'l}{.1668\,Ch^4}$$

— " —



C. E. Art. 353.

Moment of resistance to wrenching in a beam of circular cross-section.

Let r = variable distance from centre to any element; $r\,dr\,d\theta$ = area of that element; f = stress on outside fibre; $\frac{h}{2}$ = radius of circle of cross-section. Then stress at the element $r\,dr\,d\theta = f\cdot\frac{r}{\frac{h}{2}}$. Lever arm of this stress about $O = r$. Hence total mom.t is

$$M = \int_0^{2\pi}\int_0^{\frac{h}{2}} f\cdot\frac{2r}{h}\cdot r\,dr\,d\theta\cdot r$$

$$= \iint \frac{2fr^3\,dr\,d\theta}{h} = \int_0^{2\pi}\left[\frac{2fr^4}{4h}\right]d\theta = \pi f\cdot\frac{r^4}{h} = \frac{1}{16}\pi f h^3 = .196\,f h^3$$

— " —

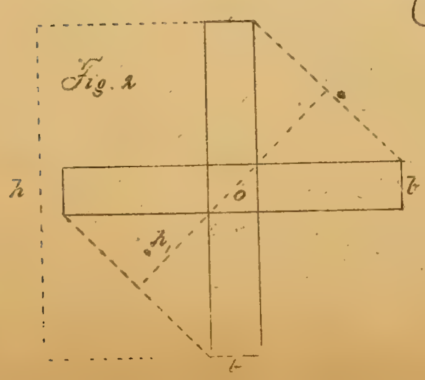
C. E. Art. 362.

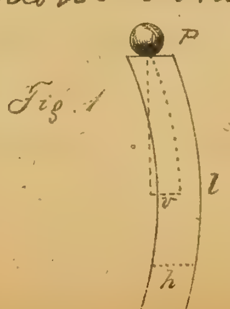
Proportion of length to diameter of screw.

Let d = diam.r; then pitch = $\frac{d}{5}$ & breadth of thread = $\frac{d}{10}$. Length of thread on a piece of bolt, = in length to $\frac{d}{2}$, is = $\frac{5}{2}\pi d = 2\frac{1}{2}\pi d$. $\therefore$ surface of thread in action = $\frac{2\frac{1}{2}\pi d^2}{10} = \frac{1}{4}\pi d^2$. Area of cross-section of spindle of bolt = $\frac{1}{4}\pi d^2$.
The latter should not, of course, exceed the former. Hence the rule.

— " —

C. E. Art. 365
(Gordon's Formula.)





The effect of a load on a pillar may be divided into two parts—
1°. A direct thrust giving a uniform comp.n on

the fibres $= \frac{P}{S} = p'$ (where P = load & S = area of cross-section)

2° A bending mom.^t causing the pillar to yield in the direction of its least dimension (h). This bending mom.^t at the point of max. strain may be represented by the expression Pv where v = the max. deviation of the pillar from a straight line. Let the stress on the external fibres due to this cause be $= p''$. Then, since

$$M = np''bh^2 ,$$

$$p'' = \frac{M}{nbh^2} = \frac{Pv}{nbh^2} \quad \therefore \quad p'' \propto \frac{Pv}{bh^2}$$

But the greatest safe deflection (Cor. 3. Deflection)

$$v \propto \frac{l^2}{h} \quad \therefore \quad p'' \propto \frac{Pl^2}{bh^3} \propto \frac{Pl^2}{Sh^2} \propto p' \cdot \frac{l^2}{h^2}$$

Hence total stress on fibres most strained is

$$\left.\begin{aligned} f &= p' + p'' = \frac{P}{S}\left(1 + a \cdot \frac{l^2}{h^2}\right) \\ \text{or} \quad P &= \frac{fS}{1 + a \cdot \frac{l^2}{h^2}} \end{aligned}\right\} \text{- - - - - - - - - - (1)}$$

where $\underline{a}$ is a constant to be found experimentally. In applying this formula to cast iron struts of the cross-section of Fig. 2. the dimension $\underline{h}$ may be measured in the direction h_1 in the figure; & the same value for $\underline{a}$ (1/500) be used as in hollow cast iron cylinders.

But it will render the formula more general &, Prof. Rankine thinks, more "satisfactory" to introduce in the place of h (measured as shown in Fig. 2. $= h_1$) the radius of gyration. The value of this radius is $r^2 = \frac{I}{\text{mass}} = \frac{n'bh^3}{n_1 bh} = \frac{n'}{n_1} \cdot h^2$, & its introduction is virtually to separate certain factors n' & n_1 which depend on the form of the cross-section. These factors are embraced in $\underline{a}$ in eq. 1. &, as they are variable with cross-section, they will render $\underline{a}$ variable in the same way. When eliminated by introducing r, the remaining part of $\underline{a}$ may be placed $= \underline{a'}$ & the formula will then become

$$f = \frac{P}{S}\left(1 + a' \frac{l^2}{r^2}\right) \quad \text{or} \quad P = \frac{fS}{1 + a' \frac{l^2}{r^2}} \quad \text{- - - - - - - (2)}$$

The mom^t of inertia of an assemblage of rectangles $= \Sigma bh^3$ (when the axis passes thro. the C. of G. of all of them). In Fig. 2.

$$\Sigma bh^3 = I = \frac{1}{12}bh^3 + \frac{1}{12}hb^3 - \frac{1}{12}b^4$$

(The central square being counted twice, we subtract its mom^t of inertia) So, area $= 2bh - b^2$

$$\therefore\ r^2 = \frac{1}{24}h^2 + \frac{1}{48}bh + \&c = \frac{1}{24}h^2 \text{ nearly.}$$

For hollow cylinders (A. M. Art. 95)

$$I = \frac{1}{64}\pi(h^4 - h'^4). \quad \text{Area} = \frac{\pi}{4}(h^2 - h'^2)$$

$$\therefore\ r^2 = \frac{1}{16}(h^2 + h'^2) = \frac{1}{8}h^2 \text{ nearly,}$$ when the cylinder is thin. Hence for hollow cast iron cylinders (in which case $\underline{a}$ as deduced from Mr. Hodgkinson's experiments $= 1/500$) we can use either

$$(1)\quad P = \frac{fS}{1 + \frac{1}{500}\cdot\frac{l^2}{h^2}} = P = \frac{fS}{1 + \frac{1}{4000}\cdot\frac{l^2}{r^2}}$$

But for a + section we have

$$P = \frac{fS}{1 + \frac{3}{500}\cdot\frac{l^2}{h^2}} = P = \frac{fS}{1 + \frac{1}{4000}\frac{l^2}{r^2}}$$

where l and h are the same throughout, but the r's vary as above shown.

In the case of the hollow square strut whose diag. $= \underline{h'}$ = diam. of the hollow cylinder, substitute in Case II. (Art. 366) for ($\underline{h}$) the side of a square its value in terms of the diag. Then

$$h = \frac{h'}{\sqrt{2}} \text{ and } r^2 = \frac{h'^2}{12}$$

In the cylinder

$$r^2 = \frac{h'^2}{8}$$

Hence

$$P = \frac{fS}{1 + \frac{3}{1000}\cdot\frac{l^2}{h'^2}} = P = \frac{fS}{1 + \frac{1}{4000}\cdot\frac{l^2}{r^2}}$$

In the case when the hollow square has its side $= h$ = diam of cylinder

For square $- r^2 = \frac{h^2}{6}$. For Cylinder $- r^2 = \frac{h^2}{8}$

Hence

$$P = \frac{fS}{1 + \frac{3}{2000}\cdot\frac{l^2}{h^2}} = P = \frac{fS}{1 + \frac{1}{4000}\cdot\frac{l^2}{r^2}}$$

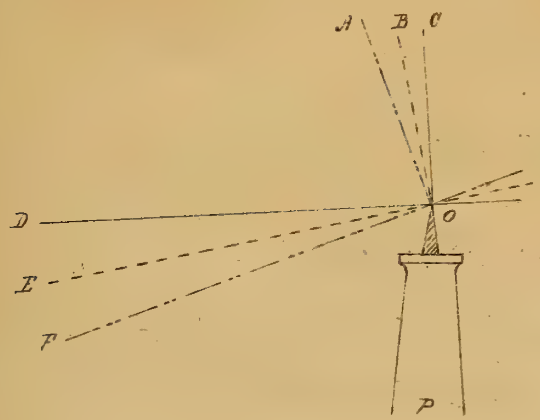
Note — Baker & others think that it is not worth while to make a change in the formula for hollow cylindrical & hollow square cast iron pillars. So in the next to the last case they would use

$$P = \frac{fS}{1+\frac{1}{500}\cdot\frac{l^2}{\frac{h'^2}{2}}} = \frac{fS}{1+\frac{2}{500}\cdot\frac{l^2}{h'^2}}$$

and in the last case $\quad P = \dfrac{fS}{1+\frac{1}{500}\cdot\frac{l^2}{h^2}}$.

— " —

C. E. Art. 372



Let PO be a pier. Also DO the tang. at O to a perfectly continuous beam; EO the tang. to an imperfectly continuous beam, in which M_0' & M_1 are to be equal; FO the tang. to a free beam: all loaded alike. Draw AO, BO, CO respectively perp. to these tang.s Then AOC = DOF = ½θ. Now find the slopes of DO, EO & FO from Art. 178. C.E.

Thus if the beam be free, M_1 in the value for the slope is = 0 &

$$\text{Slope} = \frac{1}{3EI}(w+w')c^3. \text{ (For free beam at O)}$$

If $M_1 = \frac{w+w'}{16} \cdot l^2 = \frac{w+w'}{4} \cdot c^2$; by substitution we have

$$\text{Slope} = \frac{1}{12EI}(w+w')c^3 \text{ For imperfectly continuous beam at O.}$$

If $M_1 = \frac{2w+w'}{24} \cdot l^2 = \frac{2w+w'}{6} \cdot c^2$,

$$\text{Slope} = \frac{w'c^3}{6EI}. \text{ For a perfectly continuous beam at O.}$$

The angle DOF $= \frac{\theta}{2}$ = diff. of slopes of DO & FO

$$= \frac{1}{3EI}\cdot(w+w')c^3 - \frac{w'c^3}{6EI} = \frac{1}{6EI}(2w+w')c^3$$

So the angle $EOF = \frac{1}{3EI}(w+w')c^3 - \frac{1}{12EI}(w+w')c^3 = \frac{1}{4EI}(w+w')c^3$

This angle EOF is half the contraction that has to be made to bring the free beam into the proposed condition of imperfect continuity. Hence

$$\frac{\theta}{2} : EOF :: \frac{1}{6EI}(2w+w')c^3 : \frac{1}{4EI}(w+w')c^3$$

$$\therefore\ 2EOF = \frac{3}{2}\cdot\frac{w+w'}{2w+w'}\cdot\theta$$

Subtract this from θ & the angle of the wedge to be inserted is $\theta - \frac{3}{2}\cdot\frac{w+w'}{2w+w'}\cdot\theta = \frac{w-w'}{4w+2w'}\cdot\theta$

—"—

C.E. Art. 374

Case I.

(See Corollary to Prob II - Art. 180.)

Case IV

See Cor. 1 to Prob. VI : Art. 180)

—"—

C.E. Art. 377.

III - <u>Warren</u> <u>Girder</u>.

Case. I.

The total load, when both the rolling & permanent loads are on, is

$$W = (w+w')(N-1) \quad \text{- - - - - - - - - - - (1)}$$

Reaction of the abutments (= shearing force bet. (0) & (1) Fig. 1), when whole load is on, is

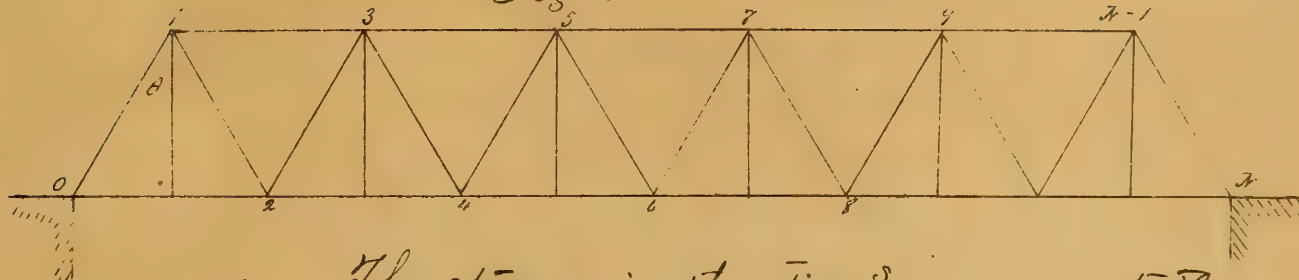
$$F_0 = \frac{W}{2} = (w+w')\cdot\frac{N-1}{2}$$

Shea<u>ri</u>ng fo<u>r</u>ce

The permanent load being uniformly distributed, the shearing force in each bay due to it is as follows -

$$\left.\begin{array}{lll}
\text{In } (0.2) & F_0' = w\frac{N-1}{2} = \text{reac. of abut}^t \text{ due to perm}^{nt}\text{ load} \\
\text{,, } (1.3) & F_1' = w\frac{N-1}{2} - w = F_0' - w \\
\text{,, } (2.4) & F_2' = w\frac{N-1}{2} - 2w = F_1' - w \\
\text{,, } (3.5) & F_3' = w\frac{N-1}{2} - 3w = F_2' - w \\
\text{,, } n^{th} & F_{n-1}' = w\frac{N-1}{2} - (n-1)w
\end{array}\right\} \text{..(3)}$$

Fig. 1



The stresses in the diag.s are equal to the shearing forces corresponding to those diag.s mult.d by the secant of the angle the diag. makes with the vert. Thus the stress due to the permanent load is -

$$\left.\begin{array}{lll} \text{In diag } (0.1) & T_0 = F_0' \cdot \frac{s}{k} = \frac{sw}{k} \cdot \frac{N-1}{2} & \\ \text{" " } (1.2) & T_1 = (F_0' - w)\frac{s}{k} = T_0 - \frac{sw}{k} & \\ \text{" " } (2.3) & T_2 = (F_1' - w)\frac{s}{k} = T_1 - \frac{sw}{k} & \\ \text{" " } n^{th} \text{ (bet. points } n \text{ \& } n+1) & & \\ & T_n = T_{n-1} - \frac{sw}{k} & \end{array}\right\} \text{---} (4)$$

This stress is compression in the diag.s (Fig. 1) that slope upwards towards the centre (like (0.1) &c) & tension in the others.

For the passing load the shearing force is a max. at any section when the longer segment into which that section divides the girder is loaded. When the shorter segment is loaded, the shearing force produces its greatest effect in reversing the stress due to the permanent load in the diag. at the section in question. Thus if only point (1) be loaded with w', the shearing force bet. (1) & (N) is $\frac{1}{N}(w')$, & this force produces a comp.n in (1.2) &c $= \frac{s}{k}\left(\frac{w'}{N}\right) = S_1$ ---------------------- (5)
which is the greatest tendency to comp.n that can ever come on (1.2) since the per.nt load, as well as the passing load when on the other joints, tends always to stretch (1.2).

Suppose joints (1) & (2) to be loaded: then the vert. shearing force bet. (2) & N due to the load at (2) is $\frac{2}{N}\cdot w'$, & the tens.n in (2.3) due to the loads at (1) & (2) is
$$= \frac{s}{k}\cdot\frac{w'}{N} + 2\cdot\frac{s}{k}\cdot\frac{w'}{N} = S_2 \text{ ---------------- } (6)$$

Load joint (3) This last load produces a shearing force bet. (3) & N = $\frac{3}{N} w'$, & the total effect on (3.4) of the loads on joints (1)(2)(3) is

$$\frac{s}{k}\cdot\frac{w'}{N} + 2\frac{s}{k}\cdot\frac{w'}{N} + 3\cdot\frac{s}{k}\cdot\frac{w'}{N} = S_3 \quad \text{-------------} (7)$$

Hence by analogy

$$\left.\begin{aligned} S_4 &= \frac{s}{k}\cdot\frac{w'}{N} + 2\frac{s}{k}\cdot\frac{w'}{N} + 3\frac{s}{k}\cdot\frac{w'}{N} + 4\cdot\frac{s}{k}\cdot\frac{w'}{N} \\ S_5 &= \frac{s}{k}\cdot\frac{w'}{N}\left(1+2+3+4+5\right) \\ &\quad \&c \qquad \&c \end{aligned}\right\} \text{----} (8)$$

The last term S_{N-1} or the stress in the last diag. is = the reaction of the abutment due to the whole pass.^{g} load mult.^{d} by $\frac{s}{k}$. or

$$S_{N-1} = \frac{s}{k}\cdot\frac{N-1}{2}\cdot w' \quad \text{-------------} (9)$$

The stress on the first diag. (0.1) is of course $S_0 = 0$, when the load is about entering the bridge.

After we pass the centre of the girder – i.e. when the longer segment is loaded – note that the pass.^{g} load produces the same kind of stress in the diag. at its end as does the permanent load. Thus when only the joints (1)(2)(3) are loaded, the action of the pass.^{g} load tends to compress (3.4) while the permanent load stretches it. But when the joints up to & including (7) are loaded, then the diag. (7.8) is compressed by the pass.^{g} as well as the permanent load.

Making then a table of the stresses due to the load as it passes ~~ov~~ over the bridge & arranging them in two columns beginning at the ends & counting towards the centre, we have

$$\left.\begin{array}{ll} S_0 & S_{N-1} \\ S_1 & S_{N-2} \\ S_2 & S_{N-3} \\ S_3 & S_{N-4} \\ \text{----} & \text{----} \\ S_n & S_{N-n-1} \\ \&c & \&c \end{array}\right\} \text{-----} (10)$$

The first of the columns gives the stresses produced when the shorter segment is loaded & they are opposite in kind to those due to the permanent load & are the greatest of their kind produced by the passing load; while the second column gives the max. stresses produced when the longer segment is loaded & they are to be added to those produced in the respective diag.^s by the permanent load, to get the total stresses. As the load may come from either end of the bridge, the columns may apply respectively to either half. The max. total stress of either comp.^n or tens.^n that may come on any diag. from the action of both loads is obtained thus. :-

1° The total max. stress in any diag. of the kind produced by the permanent load is

$$= T_n + S_{r-n-1} \text{ - - - - - - - - - } (11)$$

2° The stress produced in any diag. by the passing load when it covers the shorter segment being diff. in kind from that due to the permanent load, the total resultant in this case is

$$= T_n - S_n .$$

So long as T_n is the greater, the resultant stress is of the same kind as that above discussed; & it need not be considered, since eq. (11) gives the max. stress of that kind. But, if S_n be the greater, then we must provide for a stress in the given diag. of an opposite kind to that given by eq. (11) & of an amt.

$$= S_n - T_n \text{ - - - - - - - - - - - } (12)$$

This is done by counter bracing, or by so proportioning the brace as to fit it to bear this stress as well as that in eq. (11)

Horizontal Stresses - The max. hor. stresses occur when the full load is on the girder. In this case the shearing forces in the diff. bays are

$$\left.\begin{aligned} F_0 &= \frac{W}{2} = (w+w')\frac{r-1}{2} \\ F_1 &= (w+w')\frac{r-1}{2} - (w+w') = F_0 - (w+w') \\ F_2 &= (w+w')\frac{r-1}{2} - 2(w+w') = F_1 - (w+w') \\ &\&c \qquad\qquad \&c \end{aligned}\right\} \text{ - - - - } (13)$$

and the stresses in the diag.^{s} are then

$$\left.\begin{array}{l} F_0 \cdot \frac{s}{k} \\ F_1 \frac{s}{k} \\ F_2 \cdot \frac{s}{k} \\ \&c \end{array}\right\} \text{- - - - - - - - - - - - - - - - -} (14)$$

These last stresses mult.^{d} by $\sin\theta$ or, what is the same, the forces in (13) mult.^{d} by $\tan\theta$ give the increments of the chord stresses at each vertex.

But $\tan\theta = \frac{s}{k} \cdot \sin\theta = \frac{s}{k} \cdot \frac{Z}{Ns} = \frac{Z}{Nk}$. Hence the tens.^{n} in (0.2) is

$$= H_1 = F_0 \frac{Z}{Nk}.$$

In the 2nd bay (1.3), the 2nd diag. produces comp.^{n} as well as the 1st (0.1). The effect of this 2nd diag. is $= F_1 \frac{Z}{Nk}$

Hence the total comp.^{n} in the second bay is

$$H_2 = H_1 + F_1 \frac{Z}{Nk} \text{- - - - - - - - - - - - -} (15)$$

So in the 3^d bay (2.4) the hor. comp.^{t} of the stress in (2.3) has to be added to H_2 to get the total stress, or

$$H_3 = H_2 + F_2 \frac{Z}{Nk} \text{- - - - - - - - - - - -} (16)$$

To obtain the stress $H_{\frac{N}{2}}$ at the middle of the girder by mom.^{ts}.

Take the mom.^{s} around (6) in Fig. 1. Then

$$H_{\frac{N}{2}} k = F_0 \frac{Z}{2} - \frac{N-2}{2}(w+w')\frac{Z}{4}.$$

$$= \frac{N-1}{4} \cdot Z \cdot (w+w') - \frac{N-2}{8} \cdot Z \cdot (w+w') = \frac{NZ}{8}(w+w')$$

$$\therefore H_{\frac{N}{2}} = \frac{NZ}{8k}(w+w') \text{- - - - - - - - - - - - - -} (17)$$

So the eq. for the n^{th} bay is

$$H_n k = \frac{N-1}{2} \cdot (w+w')\frac{Z}{N} \cdot n - (n-1)(w+w')\frac{Z}{N} \cdot \frac{n}{2}$$

$$= \frac{Z}{N}(w+w')\frac{n(N-n)}{2}$$

$$\therefore H_n = \frac{Z}{Nk}(w+w')\frac{n(N-n)}{2} \text{- - - - - - - - - -} (18)$$

Case II (See Fig. 2)

Here the total load, when the pass.^{r} load extends over the girder is

$$W = (w+w')\frac{N-2}{2} \text{- - - - - - - - - - - -} (19)$$

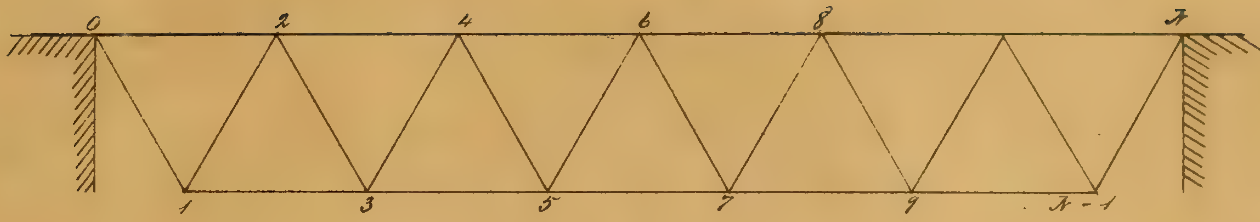


Fig. 2

and the reac.ⁿ of the abutment $F_0 = \frac{W}{2} = (w+w')\frac{N-2}{4}$ - - - - - - - - (20)

-Shearing force for permanent load-

$$\left.\begin{array}{lll} \text{In } (0.2) & F'_0 = w.\frac{N-2}{4} \\ \text{" } (1.3) & F'_1 = w.\frac{N-2}{4} = F'_0 \\ \text{" } (2.4) & F'_2 = w.\frac{N-2}{4} - w = F'_0 - w \\ \text{" } (3.5) & F'_3 = F'_2 \end{array}\right\} \text{- - - - - - -(21)}$$

Whence the stresses in the diag.ˢ due to the permanent load are

$$\left.\begin{array}{lll} \text{In } (0.1) & T_0 = F'_0 \cdot \frac{S}{k} = \frac{Sw}{k} \cdot \frac{N-2}{4} \\ \text{" } (1.2) & T_1 = F'_1 \cdot \frac{S}{k} = \frac{Sw}{k} \cdot \frac{N-2}{4} \\ \text{" } (2.3) & T_2 = F'_2 \cdot \frac{S}{k} = \frac{S}{k}(F'_1 - w) = T_1 - \frac{Sw}{k} \\ \text{" } (3.4) & T_3 = F'_3 \cdot \frac{S}{k} = \frac{S}{k}(F'_1 - w) = T_1 - \frac{Sw}{k} \\ & \text{\&c} \quad \text{\&c} \end{array}\right\} \text{- -(22)}$$

From this it is seen that the diag.ˢ which meet at an unloaded vertex bear stresses equal in amt. though diff. in character.

In forming the columns giving the stresses in the diag.ˢ due to the pass.ᵍ load, the values of S_0 & S_1 become equal & so of S_2 & S_3. &c. Hence eq. (10) becomes for Fig. 2.

$$\left.\begin{array}{ll} S_0 = S_1 & S_{N-1} = S_{N-2} \\ S_2 = S_3 & S_{N-3} = S_{N-4} \\ S_4 = S_5 & S_{N-5} = S_{N-6} \end{array}\right\} \text{- - - - - (23)}$$

When $\frac{N}{2}$ is odd there is but a single value at the foot of each column. In Fig. 3. it is

S_6 $\qquad$ S_{N-7}

The total stress due to both loads is as before the

sum of the T & S belonging to the brace in question. Thus the total max. stress in (3.4) Fig. 2. when the longer segment is loaded is

$$T_3 + S_{N-4} = T_2 + S_{N-3} = \text{stress in } (2.3) \quad \text{---} (24)$$

When the shorter segment is loaded, the stress is

$$S_3 - T_3 = S_2 - T_2$$

& if S be the greater, provision must be made for a stress of the above amt. of an opposite kind to that given by eq. (24)

Horizontal stresses - Proceeding as before, we have for compr. in (0.2)

$$\left.\begin{array}{ll} & \mathcal{H}_1 = \frac{l}{Nk}\cdot \mathcal{F}_0 = \frac{l}{Nk}\cdot \mathcal{F}_1 \\ \text{In } (1.3) & \mathcal{H}_2 = \mathcal{H}_1 + \frac{l}{Nk}\cdot \mathcal{F}_1 = 2\frac{l}{Nk}\cdot \mathcal{F}_0 \\ \text{In } (2.4) & \mathcal{H}_3 = \mathcal{H}_2 + \frac{l}{Nk}\cdot \mathcal{F}_2 \\ \text{"} \ (3.5) & \mathcal{H}_4 = \mathcal{H}_3 + \frac{l}{Nk}\cdot \mathcal{F}_3 = \mathcal{H}_2 + 2\frac{l}{Nk}\mathcal{F}_2 \end{array}\right\} \quad \text{---} (25)$$

To obtain the value of $\mathcal{H}_{\frac{N}{2}}$ when $\frac{N}{2}$ is even, take mom.ts about (6) in Fig. 2. &

$$\mathcal{H}_{\frac{N}{2}} k = \frac{N-2}{4}\cdot\frac{l}{2}\cdot(w+w') - \frac{N-4}{4}\cdot\frac{l}{4}\cdot(w+w')$$

$$= \frac{Nl}{16}(w+w')$$

$$\therefore \mathcal{H}_{\frac{N}{2}} = \frac{Nl}{16k}(w+w') \quad \text{---} (26)$$

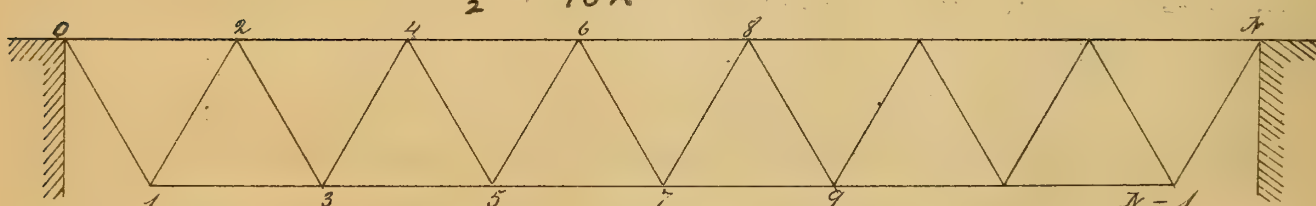


Fig. 3

When $\frac{N}{2}$ is odd (Fig. 3.) take mom.ts about (7)

Then $\mathcal{H}_{\frac{N}{2}} k = \frac{N-2}{4}\cdot\frac{l}{2}(w+w') - \frac{N-2}{4}\cdot(w+w')\frac{N+2}{4}\cdot\frac{l}{N}$

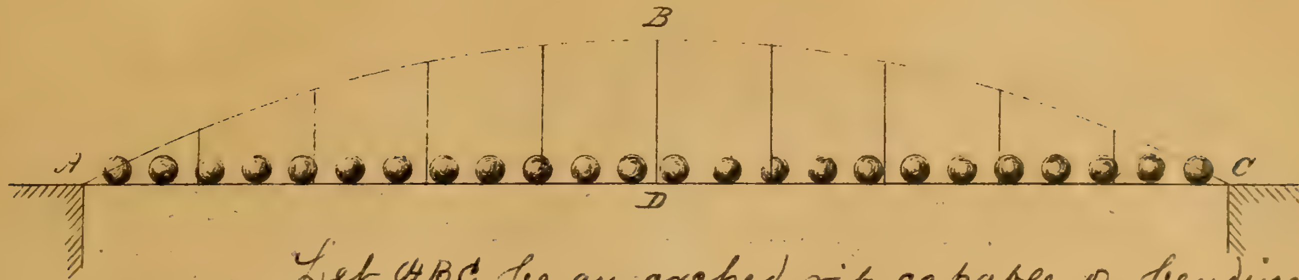
$$= \frac{N-2}{4}\cdot l(w+w')\left(\frac{1}{2} - \frac{N+2}{4N}\right) = \frac{(N-2)^2}{16N}\cdot l\cdot(w+w')$$

$$\therefore \mathcal{H}_{\frac{N}{2}} = \frac{(N-2)^2}{16Nk}\cdot l\cdot(w+w') \quad \text{---} (27)$$

C. E. Art. 379.

(Bow String Girder)

Fig. 1



Let ABC be an arched rib capable of bending so as to assume the curve of equilibrium, & let it be held at A & C either by abutments or as in the case of a Bow String Girder by a tie AC. When the load on such a rib (whether the load be placed on the rib itself or be suspended from it by vert. rods) is distributed uniformly along the horizontal, the rib AC becomes a parabola; this being the curve of equilibrium for such a load. Under such a load there is no necessity for diag. bracing, the purpose of which is to resist the tendency of the rib to change shape under loads not distributed uniformly along the horizontal, as in the case of a girder partly loaded with a passing train.

Let w = permanent load or wt. of girder per running foot.

Let w' = wt. of passing load per foot.

" l = length of girder (= AC).

" h = depth (BD) at the crown.

The greatest total load is when the passing load covers the whole bridge. Then

$$W = (w + w')l$$

To determine the thrust H at the crown or the equal tens.^n at D in the tie produced by this load. Imagine the girder to be cut at BD by a vert. plane. Then the forces under which the half girder ABD is balanced are

(1) The reaction of the abutment (= R).

(2) The load bet. A & B $(= \frac{W}{2})$.

(3) The thrust H at B. (4). The tens.^n at D.

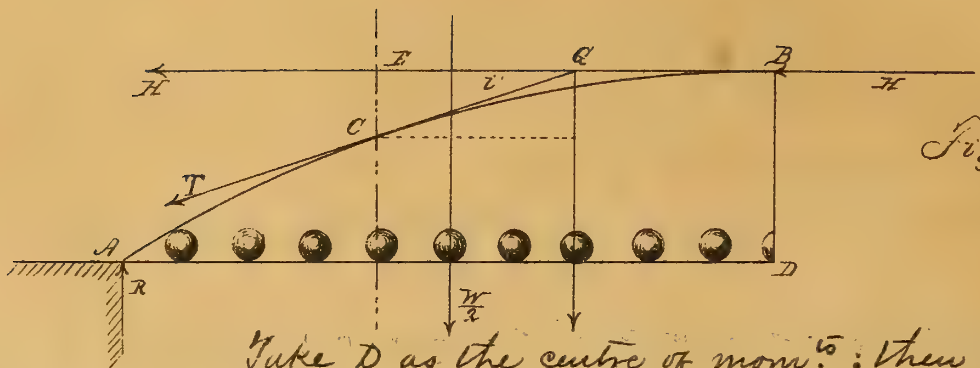


Fig. 2

Take D as the centre of mom.ts; then the sum of the mom.ts of the forces about D must =

$$\therefore R \cdot \frac{l}{2} - \frac{W}{2} \cdot \frac{l}{4} - Hk = 0$$

But

$$R = \frac{(w+w')l}{2} \text{ and } \frac{W}{2} = \frac{(w+w')l}{2}$$

$$\therefore Hk = \frac{w+w'}{2} \cdot \frac{l^2}{2} - \frac{w+w'}{2} \cdot \frac{l^2}{4} = \frac{(w+w')l^2}{8}$$

$$\therefore H = \frac{w+w'}{k} \cdot \frac{l^2}{8} \text{ - - - - - - - - - - - - (1)}$$

By taking B as the centre of moments we find the tension at D in the tie $AC = H$. This tens.n is uniform throughout the tie.

The thrust along the rib at any point C is the resultant of the thrust H at the crown & of the load bet. B & C. For, considering the section CB, of the girder cut out by the two planes at C & B (Fig. 2), it is evident that the thrust at C must be equal & opposite to the force H & the load $(w+w')x$ (where x = hor. distance from B to C). Let the thrust at $C = T$.

Then

$$T = \sqrt{H^2 + (w+w')^2 x^2} = H \sec i$$

At A, this is

$$T_1 = \sqrt{H^2 + (w+w')^2 \frac{l^2}{4}}$$

The thrust H at the crown of the rib & the tens.n in the tie &, consequently, the thrust T at any point of ABC are maxima when the entire load is on the girder. For consider the thrust H' at B due to the passing load, & let the load extend from C (Fig. 4) to E. Then, taking mom.ts around D for the half girder ADB, we have

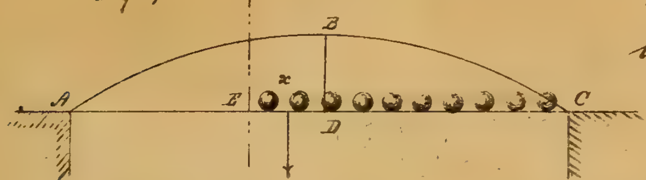
$$H'k = R'\cdot\frac{l}{2} - w'x\cdot\frac{x}{2}$$

Now
$$R' = w'\left(\frac{l}{2}+x\right)\frac{\frac{l}{2}+x}{2l} = w'\left(\frac{l}{2}+x\right)\frac{l+2x}{4l}$$

$$\therefore\ H' = \frac{w'}{k}\left[\left(\frac{l}{2}+x\right)\frac{l+2x}{8} - \frac{x^2}{2}\right]$$

Applying tests for a max.
$$\frac{dH'}{dx} = \frac{w'}{k}\left[\frac{l+2x}{8}+\frac{l+2x}{8} - x\right] = 0$$
whence
$$\frac{l}{4}+\frac{x}{2}-x = 0 \quad \text{or} \quad x = \frac{l}{2}$$

Fig 4



Hence H is a max. when the load covers the whole bridge

To determine the stresses on the bracing due to a passing load. Let the load advance from A towards C & let it be considered as concentrated at the panel divisions. Then the diag. braces, if they be ties, should slope as in Fig. 5, & the greatest stress on any one of them as EZ occurs when the load covers the girder up to & including E. Considering B as the origin & putting y = the vert. coordinate of the curve ABC, its eq. is
$$x^2 = 2py.$$
At C this becomes
$$\frac{l^2}{4} = 2pk \quad \therefore\ 2p = \frac{l^2}{4k}$$
Hence the eq. of ABC is
$$x^2 = \frac{l^2}{4k}\cdot y \quad \text{or} \quad y = \frac{4k}{l^2}\cdot x^2 \text{ - - - - - - - - (2)}$$
Draw a tang. at any vertex F. The subtang. LR is bisected at B ∴ LR = $2y$. The triangles LFR & FEO are similar. Hence, if we put EO = m, we have
$$2y : x :: k-y : m = EO = \frac{x(k-y)}{2y} = \frac{l^2-4k^2}{8x}$$ by substituting value of y from (2).

Again
$$CO = m - \left(\frac{l}{2} - x\right) = \frac{(l-2x)^2}{8x}$$
Produce the direction of the diag. ET & drop the perp. OS upon it. $OS = EO \sin\theta$

Imagine a plane of section just beyond FE. The forces acting on the righthand part of the girder are

1° Thrust at F. 2° - Tension on tie AC. 3° Tension (= D) in diag. ET. 4° Reaction of abutment at C(= R).

Taking mom.^s about O where the directions of the first two of these forces intersect, we have

$$R.\overline{CO} = D.\overline{OS}$$

$$\therefore \quad D = R.\frac{CO}{OS} = R.\frac{CO}{EO\sin\theta} = R.\frac{CO}{EO}.\operatorname{cosec}\theta \text{ ------ } (4)$$

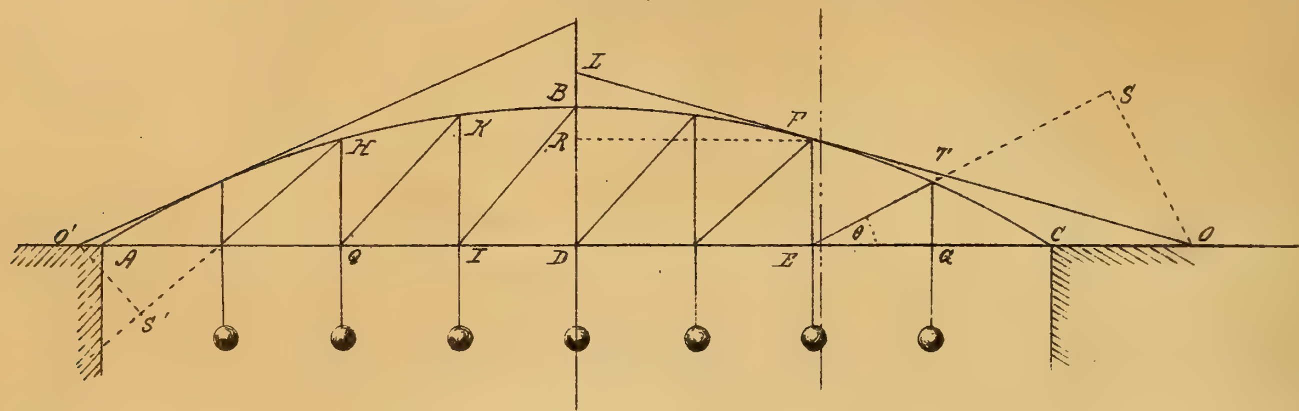


Fig. 5

Let N = number of panels in girder: $\frac{2l}{N}$ = length of one panel: n = no. of any panel & also of the vert. & diag. next beyond counting from A. Load at each vertex = $\frac{w'2l}{N}$ & the reaction at C, when the girder is loaded as in the figure is

$$R = w'n.\frac{2l}{N}.\frac{n+1}{2N}$$

Also
$$\frac{CO}{EO} = \frac{(2l-2x)^2}{l^2-4x^2} = \frac{l-2x}{l+2x}$$

$$\therefore \quad D = w'\frac{n(n+1)}{2N^2}2l.\frac{l-2x}{l+2x}.\operatorname{cosec}\theta$$

But we may consider
$$x = n\frac{2l}{N} - \frac{l}{2}$$

$$\therefore \quad l-2x = l - \frac{22ln}{N} + l = 2l.\frac{N-n}{N}$$

and
$$l+2x = l + \frac{22ln}{N} - l = 2l.\frac{n}{N}$$

$$\therefore \quad \frac{l-2x}{l+2x} = \frac{N-n}{n}$$

$$\therefore \quad D = w'\frac{n(n+1)}{2N^2}2l.\frac{N-n}{n}.\operatorname{cosec}\theta \text{ -------- } (6)$$

It will be most convenient to replace the cosec. θ also by its value. Thus -

$$\operatorname{cosec}.\theta = \frac{EI}{IC_c} = \frac{\sqrt{\frac{l^2}{N^2}+(k-y')^2}}{k-y'}$$ where y' is the ordinate of I. At I

$$y' = \frac{4k}{l^2}x'^2 \quad \& \quad x' = x + \frac{l}{N}$$

$$\therefore\ k-y' = k - \frac{4k}{l^2}\left(x+\frac{l}{N}\right)^2 = k\left[1-\frac{4}{l^2}\left(x+\frac{l}{N}\right)^2\right]$$

Since $x = n\frac{l}{N} - \frac{l}{2}$, we have

$$1-\frac{4}{l^2}\left(x+\frac{l}{N}\right)^2 = 1-\frac{4}{l^2}\left(\frac{nl}{N}-\frac{l}{2}+\frac{l}{N}\right)^2 = 1-\frac{4}{l^2}\left(\frac{2(n+1)-N}{2}\right)^2\cdot\frac{l^2}{N^2}$$

$$= 1-\frac{4(n+1)^2-4N(n+1)+N^2}{N^2} = \frac{4}{N^2}\left[(N-n-1)(n+1)\right]$$

Hence
$$\operatorname{cosec}\theta = \frac{\sqrt{\frac{l^2}{N^2}+\frac{16k^2}{N^4}\left[(N-n-1)(n+1)\right]^2}}{\frac{4k}{N^2}\left[(N-n-1)(n+1)\right]}$$

$$= \frac{N\sqrt{l^2+\frac{16k^2}{N^2}\left[(N-n-1)(n+1)\right]^2}}{4k\left[(N-n-1)(n+1)\right]}$$

Hence finally
$$D = \frac{w'l(N-n)}{2N}\cdot\frac{\sqrt{l^2+\frac{16k^2}{N^2}\left[(N-n-1)(n+1)\right]^2}}{4k\left[N-n-1\right]} \quad \text{------ (7)}$$

This formula evidently applies to every diag. bet. B & C. by giving proper values to n.

If we take a section just to the right of KQ, the girder being only loaded up to & including Q, & form the eq. of mom.^{ts} for the forces on the segment KC of the girder about the point O', we shall find that the expression for the stress in the diag. QK reduces to precisely the form given in eq. (7). Hence that formula is general for the whole girder.

A load coming on the truss from C requires diag.^{s} inclined in a direction opposite to those in the Fig.

<u>Stress</u> <u>in</u> <u>the</u> <u>verticals</u> (Fig. 6) The tens.^{n} in the chord from E to J is greater than from E to C because

the diag. EI, when the partial load is on, pulls at E. Hence leaving out that part of the tens.^{n} in the bay EJ which is neutralized by the stress in EC we have at E two pulls, one from E towards I & the other towards J, & hence the stress in FE due to the partial load must be one of comp.^{n} To determine its amt, pass a cutting

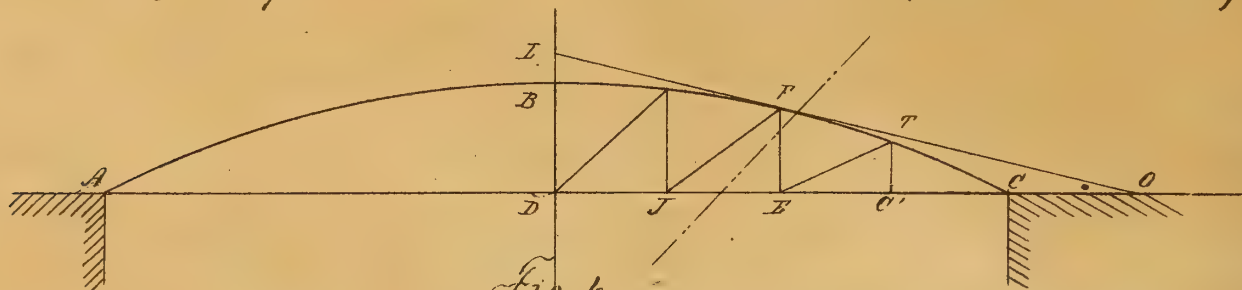


Fig. 6

plane, as in Fig. 6, just behind F & so as to cut the vert. FE & the chord AC in the bay JE. It is evident that the load at E would tend to stretch FE & hence the max. comp.^{n} in that vert. is produced when the girder is loaded from A up to, & including J.

Taking the mom.^{s} of the forces on the right-hand segment of the truss, as before, about O, we have (calling the stress in FE; V')

$$V'.\overline{EO} = R.CO.$$

$\underline{n}$ being the no. of the bay JE & of the vert. FE, we have

$$R = w'(n-1)\frac{l}{N}\cdot\frac{n}{2N} = \frac{w'n(n-1)l}{2N^2}$$

Also $\quad \frac{CO}{EO} = \frac{N-n}{n} \quad \therefore \quad V' = \frac{w'(n-1)l(N-n)}{2N^2}$ - - - - - - - (8)

This formula, like that for D, applys to all the verts.

To obtain the real comp.^{n} in any vert., we must subtract from V' the proportion of the permanent load that hangs from that vert.. The result, if positive, gives the max. comp.^{n} to be provided for. If the result be negative, the vert. never suffers comp.^{n} The greatest tens.^{n} on any vert. occurs when the whole load is on the girder & is = the pass.^{g} load on one panel _plus_ that part of the permanent load on one panel which hangs on the vert.

When the number (N) of panels is odd, the values of D & V' are the same as above.

Example

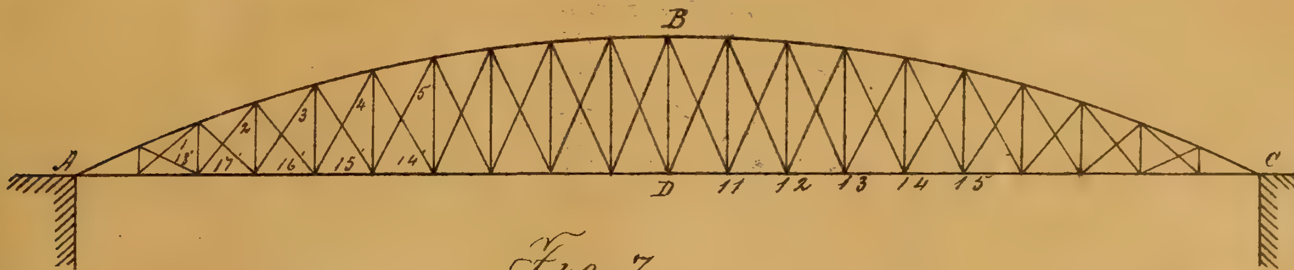


Fig. 7

Take as an example the one given by Shreve. The data are $l = 200 ft$ $k = 25 ft$
$N = 20$ $\frac{l}{N} = 10 ft$
$w = \frac{1}{2}$ ton per running ft. $w' = 1$ ton per running ft

From eq. 1. the max. thrust at the crown = max. tens.^{n} in AC also is

$$H = \frac{1+\frac{1}{2}}{25}(200)^2 \cdot \frac{1}{8} = 300 \text{ tons}$$

At A & C

$$T_1 = \sqrt{(300)^2 + 22500} = 335+ \text{ tons}$$

Substituting the successive values of n in eq. 7, we obtain the following tens.^{ns} in the diag.^{s} which are those to be provided for.

Values of n	Diag.^{s}	Tons (Tension)	By Rank.^{s} Formula
1	1 & 1'	14.20	.75
2	2 & 2'	17.15	
3	3 & 3'	20.04	
4	4 & 4'	22.66	
5	5 & 5'	24.92	8.31
6	6 & 6'	26.70	
7	7 & 7'	28.16	
8	8 & 8'	29.12	
9	9 & 9'	29.62	

Values of n	Diag.^{s}	Tons (Tension)	By Rank.^{s} Formula
10	10 & 10'	29.66	29.61
11	11 & 11'	29.25	
12	12 & 12'	28.40	
13	13 & 13'	27.14	
14	14 & 14'	25.50	
15	15 & 15'	23.58	70.80
16	16 & 16'	21.60	
17	17 & 17'	20.18	
18	18 & 18'	22.15	189.30

The last column gives some of the stresses by Rankine's formula.

For the compⁿ in the vert.ˢ, use eq.(8) taking note that we need only obtain the stresses in the verts of ½ the girder when the longer segment is loaded. The others will be strained like them when the load comes in the opposite direction.

Note too that the tensⁿ due to the permanent load is to be subtracted. The wt. of the permanent load for one panel is = 5 tons. Supposing the arch to weigh ⅓ of this, there remains ⅔ of 5 = 3.33 tons of the permanent load hanging from each vert.

Values of n	Number of Vertical	Tons Compression Pass.ᵍ Load.	Tons Tension Per.ⁿᵗ Load.	Max. Compⁿ to be provided for.	Max. Tensⁿ to be provided for.
10	10	22.50	3.33	19.17	13.33
11	11	22.50	3.33	19.17	13.33
12	12	22.00	3.33	18.67	13.33
13	13	21.00	3.33	17.67	13.33
14	14	19.50	3.33	16.17	13.33
15	15	17.50	3.33	14.17	13.33
16	16	15.00	3.33	11.67	13.33
17	17	12.00	3.33	8.67	13.33
18	18	8.50	3.33	5.17	13.33
19	19	4.50	3.33	1.17	13.33

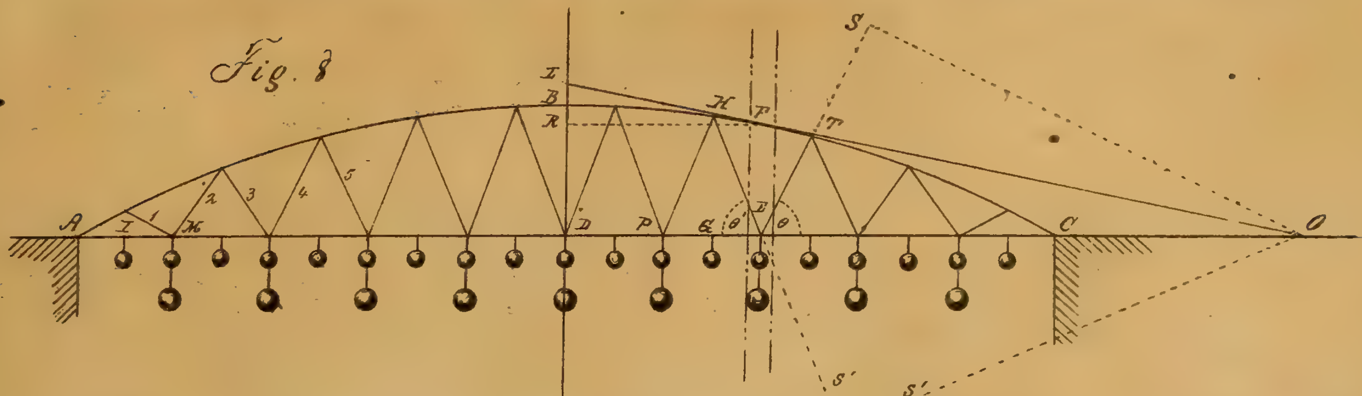
The passing load on one panel = 10 tons. Hence the max. tensⁿ on each vert. due to both loads is 13.33 tons.

From the table it is seen that all the vert.ˢ in Fig. 7 need to be adapted to bear both compⁿ & tensⁿ.

If the truss be inverted, however, this will not be the case. By inverting the girder, the bow is subjected to tensⁿ & the chord AC to compⁿ & the verts to compⁿ always. The amt. of the compⁿ in the vert.ˢ is obtained by adding the 3ʳᵈ & 4ᵗʰ columns above except where the sum is less than 13.33 in which case the latter number is to be used. The stresses in the diag.ˢ are the same as those already given.

Bow String Girder with Isosceles Bracing.

Fig. 8



1°. Action of passing load. In this case, consider the girder as divided into a number of half-panels = N. Each of these as $AI = \frac{l}{N}$. Suppose the load concentrated at these half-panel lengths, as in the figure, & let n = the number of any half-panel = also the number of the brace next beyond this half-panel counting from A. Thus, for the half-panel PQ, HE is the n^{th} brace &, if QE = n^{th} half-panel, EF is the n^{th} brace.

The braces inclined like EF are in tens.^{n} when the load comes on the bridge at A; & those inclined like HE are compressed. These stresses are reversed in character when the load comes on at C. FE suffers its max. tens.^{n} when the load extends up to, & includes E.

Pass the cutting plane as before just to the right of E, &, having drawn a tang. at F (just over E) & completed the fig. as in the last case, we have by taking mom.^{ts} around O & considering the load as concentrated at the ends of the full panels,

$$D.\overline{OS} = R.\overline{CO}$$

$$\text{Load at each vertex} = 2\frac{l}{N}.w'.$$

$$\text{Load up to section} = w'.\frac{2l}{N}.\frac{n}{2} = \frac{w'nl}{N}$$

$$R = \frac{w'nl}{N}.\frac{n+2}{2N}$$

OS & CO have the same expressions for their values as in the last case

$$\therefore\ D = \frac{w'l(n+2)}{2N}(N-n).\frac{\sqrt{l^2+\frac{16k^2}{N^2}\left[(N-n-1)(n+1)\right]^2}}{4k(N-n-1)(n+1)} \quad \text{- - - - - - - -}(9)$$

When the section is just to the left of E, HE is the n^{th} brace, & PG the n^{th} half-panel. For stress in HE

$$D' . OS' = R' . CO$$

$$D' = R' \frac{CO}{EO} \operatorname{cosec} \theta'$$

$$u = (AG + GE) - AD = n . \frac{l}{N} + \frac{l}{N} - \frac{l}{2} = \frac{l}{N}(n+1) - \frac{l}{2}$$

$$\text{Acting load} = w' . \frac{n-1}{2} . 2\frac{l}{N} = \frac{w'(n-1)l}{N}$$

$$\therefore R' = \frac{w'(n+1)(n-1)l}{2N^2}; \qquad \frac{CO}{EO} = \frac{l-2u}{l+2u} = \frac{N-n-1}{n+1}$$

$$\operatorname{cosec} . \theta' = \frac{N\sqrt{l^2 + \frac{16k^2}{N^2}[(N-n)n]^2}}{4nk(N-n)}$$

$$\therefore D' = \frac{w'l(n-1)(N-n-1)}{2N . 4kn(N-n)} . \sqrt{l^2 + \frac{16k^2}{N^2}[n(N-n)]^2} \text{------} (10)$$

This formula applies to all braces inclined like HE when the ~~las~~ load comes on at A. Eq. (10) gives the stress when n is odd, & the stress is comp.^{n}; while eq. (9) gives the value when n is even & the stress is tension.

2° Action of a permanent load. Under such a load the bow is equilibrated. The tie AC suffers a uniform tens.^{n} = in amt. to the thrust H at the crown of the bow & the bracing merely transmits the loading on the hor. chord to the arched rib. The stress thus produced on the braces is tensile; the sum of the vert. comp.^{ts} of the stress in any two braces meeting in a point of the lower chord (as HE & IE) is equal to the load suspended there, equal to the load on one panel-length. Let this load = W. The hor. comp.^{s} of the tens.^{n} in HE & IE are necessarily equal, since the stress in the hor. chord on each side of E is the same (this last can readily be proven by mom.^{s})

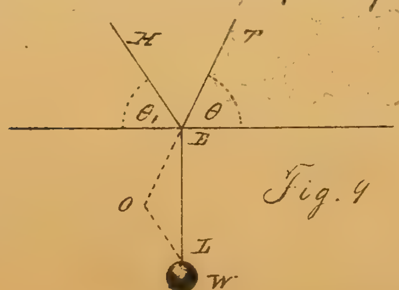


Fig. 9

Hence, if we lay off EL = W & draw EO & OL (Fig. 9) parallel to the braces, the triangle EOL gives the relation bet. the forces. Let D_1 = tens.^{n} in EI & D_1' = that in HE;

then $W : D_1 : D_1' :: \sin(\theta+\theta') : \cos\theta' : \cos\theta$

$$\therefore\ D_1 = \frac{W\cos\theta'}{\sin(\theta+\theta')} \quad ; \quad D_1' = \frac{W\cos\theta}{\sin(\theta+\theta')} \quad \text{-----} (11)$$

The stresses in the diag.^s differ because the angles θ & θ' are different.

Example

$l = 200$ ft. $\qquad N = 20$ half-panels (Fig. 8)
$k = 25$ ft $\qquad \frac{l}{N} = 10$ ft
Permanent load = ½ ton per running foot
Passing " = 1 " " " "

Use formulae (9) & (10) to determine the stresses due to the passing load, using (9) when $n = 2, 4, 6$ &c, & (10) when $n = 1, 3, 5$ &c. The following table gives the result. This table gives the comp.^n & tens.^n that may come on each brace: Those that are stretched when the load comes on at A being compressed when it comes on at C.

Values of n	No. of brace	Tons Comp.^n	Tons Tens.^n
1	D_1	00.00	23.31
3	D_3	10.16	22.87
5	D_5	15.87	27.20
7	D_7	19.66	30.57
9	D_9	21.57	32.36
11	D_{11}	21.57	32.36
13	D_{13}	19.66	30.57
15	D_{15}	15.87	27.20
17	D_{17}	10.16	22.87

Values of n	No. of Brace	Tons Tens.^n	Tons Comp.^n
2	D_2	22.87	10.16
4	D_4	27.20	15.87
6	D_6	30.57	19.66
8	D_8	32.36	21.57
10	D_{10}	32.36	21.57
12	D_{12}	30.57	19.66
14	D_{14}	27.20	15.87
16	D_{16}	22.87	10.16
18	D_{18}	23.31	00.00

For the effect of the permanent load, we must know the angles made by the braces with the horizontal. Let the angles be named θ_1, θ_2 &c. Then any one of them as

$$\theta_{13} = \tan^{-1}\left(\frac{HG}{GE}\right) \tan^{-1}\left(\frac{k-y}{\frac{l}{N}}\right) = \tan^{-1}(2.275) = 66^\circ\ 16'$$

So $\theta_1 = \theta_{18} = 25° 25'$; $\theta_4 = \theta_5 = \theta_{15} = \theta_{14} = 61° 56'$

$\theta_2 = \theta_3 = \theta_{16} = \theta_{17} = 51° 57'$; $\theta_6 = \theta_7 = \theta_{12} = \theta_{13} = 66° 16'$

$\theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 68°$

Suppose 2/3 of the permanent load to hang from the braces. This = 6.66 tons at each vertex. Then, using formula (11), the stress on any brace, as HB, is

$$D_{13} = 6.66 \cdot \frac{\cos\theta_{14}}{\sin(\theta_{13}+\theta_{14})}$$

Number of Braces.	Tension due to Per.t Load	Max. Compress.n	Max. Tension.
$D_1 = D_{18}$	4.22	0.00	27.53
$D_2 = D_{17}$	6.17	3.99	29.04
$D_3 = D_{16}$	3.43	6.73	26.30
$D_4 = D_{15}$	4.50	11.37	31.70
$D_5 = D_{14}$	3.41	12.46	30.61
$D_6 = D_{13}$	3.99	15.67	34.56
$D_7 = D_{12}$	3.48	16.18	34.05
$D_8 = D_{11}$	3.75	17.82	36.11
$D_9 = D_{10}$	3.59	17.98	35.95

The tens.n due to the permanent load is a constantly acting force & consequently, the true max. stresses to be provided for are obtained by taking the algebraic sum of this tens.n & the stresses in the preceding table due to the pass.g load. The greatest stresses in the chords are:–

$$H = \frac{1\frac{1}{2}}{25}\,\frac{(200)^2}{8} = 300 \text{ tons.}$$

$$T_1 = 335+. \text{tons.}$$

— " —

Lenticular Truss

When such a truss is uniformly loaded all over, we have the thrust H at B or tens.n H′ at B′ given by the eq.

$$H = \frac{1}{8}\,\frac{(w+w')l^2}{d} = H'$$

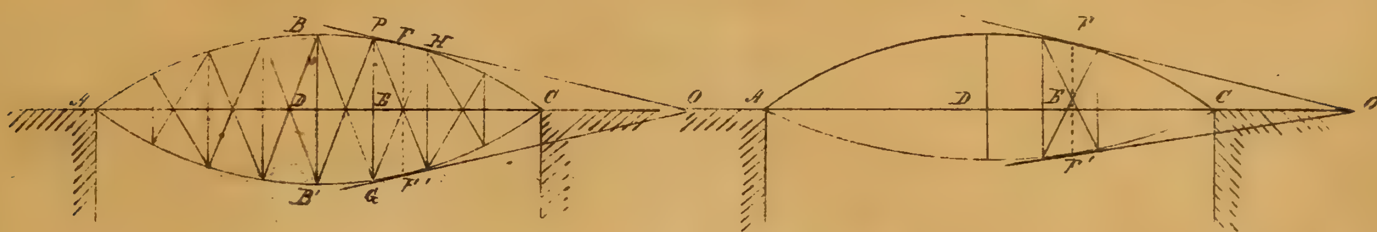
where $d = BD + B'D = k + k'$

If the versines are equal or $k = k'$,

$$H = \frac{1}{8k}\left(\frac{w+w'}{2}\right)l^2 = H' \quad \text{or}$$

the stress is half what it would be in a Bow String Girder

Fig. 1 Fig. 2



whose depth $= k$ & length $= l$.

The load is supported, ½ by the bow & ½ by the tie. The latter is transmitted to the tie by the vert. struts. If $\frac{l}{N}$ = length of a panel, then each vert. bears a wt. $= \frac{w'}{2}\cdot\frac{l}{N} + w''$ when the full load is on. Here w' = wt. of passing load per ft, & w'' = wt. per ft. of that part of the permanent load which rests on the verticals.

For the stresses in the diag.s under a passing load (they are not strained under a full load) draw tang.s at F & F', the points of the chords corresponding to E where the braces intersect; & it is evident that for the stress in any tie as HG we have the same expression as in the Bow String Girder, eq. (5)

$$D = w'\frac{n(n-1)}{2N^2}l\cdot\frac{l-2x}{l+2x}\cdot\operatorname{cosec}\theta \text{ - - - - - } (1)$$

It is simpler to leave the eq. in this form, as to substitute for x & cosec. θ in terms of n gives a complicated expression.

So, the stress in the vert.s as EG is

$$V = w'' + w'(n-1)\frac{l}{N}\cdot\frac{n}{2N}\cdot\frac{l-2x}{l+2x} \text{ - - - - - - - } (2)$$

When the versines are different (the case in the text) or k not equal to k' – Let W = wt. of bow; W' = wt. of tie. Then, when the truss bears only its own wt.

$$Hk = \frac{1}{8}Wl \quad \text{and} \quad H'k' = \frac{1}{8}W'l$$

Since H always $= H'$ $\quad k : k' :: W : W'$

There is no stress in the vert.s in this case unless the floor is fastened to them along the line ADC; in which case the wt. of the floor should be included in W'.

The stresses in the vert.^{s} & diag.^{s} due to the passing load are obtained as before. For the value of EO being $= \frac{l^2 - 4k^2}{8k}$ & ∴ not involving k, the tang.^{s} at F & F' will intersect at the same point on AC prolonged as if k had been $= k'$.

C. E. Art. 380.
(Case. I.)

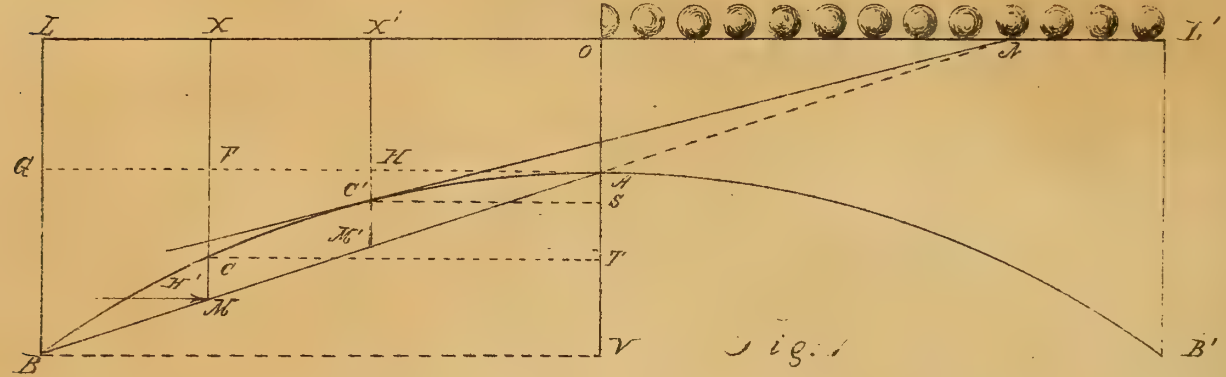


Fig. 1

Eq.^{s} (1), (2), (3) are the usual ones for thrust at the crown. When the beam is half loaded as in the fig. taking mom.^{ts} for the left half of the girder around V we have

$$H'k = R \cdot \frac{l}{2} \quad . \qquad \text{But}$$

$$R = \frac{w'l}{2} \cdot \frac{1}{4} \cdot \quad = \frac{1}{8} w'l \quad .$$

$$H' = \frac{w'l^2}{16k} = \frac{w'c^2}{4k} = \frac{H_1}{2}$$

(H' now being = thrust at crown.)

Again, considering the moving load only, the half-girder BO when the other half is loaded is acted on by but two forces – the thrust at A & the reaction of the abutment at B & these forces must ∴ be equal & opposite to each other, since the girder is balanced. Hence, they both act in the direction of the chord BA. The hor. comp.^{t} of these forces $= H' = \frac{H_1}{2}$ as already shown. The thrust at the crown in the direction AB is transmitted to the abutment by means of the curved & straight beams ACB & OL; & to determine the stresses in these beams, take any section MCX. The hor. force at $M = H' =$ hor. comp.^{t} of the force along AB. This is balanced by the hor. forces at C & X. Let that at $C = H_2$ & that at

$X = H_3$. Then

$$(H' = \tfrac{H_1}{2}) : H_2 : H_3 :: CX : MX : CM$$

$$\therefore \quad H_3 = \frac{H_1 . CM}{2 CX} \text{ - - - - - - - - - - - - (4)}$$

and

$$H_2 = \frac{H_1 . MX}{2 CX} \text{ - - - - - - - - - - - - - (5)}$$

This last is a max. when the ratio $\frac{MX}{CX}$ is a max. which evidently takes place when the angle made by the line from C to N with LN is the least possible. This occurs when NC is tang. to the curve at C'.

Algebraically - The curve ACB being a parabola, take A as origin & count x horizontally & y vertically. Then its eq. is

$$y = \frac{k}{c^2} . x^2$$

Since for any ordinate FC

$$FC : GB :: x^2 : c^2 \quad \therefore \; FC = \frac{kx^2}{c^2}$$

add $FX = a$ and $\quad CX = \frac{ac^2 + kx^2}{c^2}$

The similar triangles AGB & AFM give

$$MF : GB :: x : c \quad \therefore \; MF = \frac{kx}{c}$$

add $FX = a$ and

$$MX = \frac{ac + kx}{c}$$

$$\therefore MC = MX - CX = \frac{ac + kx}{c} - \frac{ac^2 + kx^2}{c^2} = \frac{kcx - kx^2}{c^2}.$$

Hence (4A) & (5A).

Eq. (6) is obtained by the usual tests for a max.

Eq.s (7) & (8) present no difficulty.

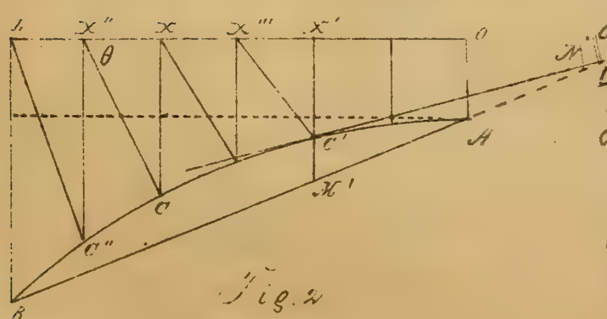


Fig. 2

To find the stress in the diag.s due to the passing load when the girder is half loaded as above. The stress in the beam OL (= H_3) diminishes from the point X' of max. stress towards L and O. In going from L towards X' the stress in that beam can only increase by reason of the horizontal components of the stresses in the diag.s, since the verts can have no effect upon a hor. force. Hence the difference bet. the stress

at X for instance & X″ (Fig. 2) in the beam OL must = the hor. compt of the stress in CX″. If x = distance OX & x' = distance OX″ this difference is

$$H_3 - H_3' = \frac{H_1}{2}\left[\frac{kcx - kx^2}{ac^2 + kx^2} - \frac{kcx' - kx'^2}{ac^2 + kx'^2}\right].$$

∴ compression in CX″ is

$$= \frac{H_1}{2}\left[\frac{kcx - kx^2}{ac^2 + kx^2} - \frac{kcx' - kx'^2}{ac^2 + kx'^2}\right] \sec.\theta \text{ - - - - - - (a)}$$

Between X′ and O the stress in the diags, if inclined in the same direction as CX″ will be tensile. It would be well to unite the curved & hor. ribs for that distance.

— " —

C.E. Art. 381.

The greatest intensity of stress on a column is found by taking the algebraic sum of the stress due to the direct compn f' & that due to the momt f''. Then

$$f = f' \pm f''$$

Now $\quad f' = \frac{P}{A}$ and from eq. (5) Art. 162 and

Case VI Art. 163, $\quad M = \frac{\pi}{32}\cdot f''\cdot\frac{h^4 - h'^4}{h}$

But $h' = h - 2t$ $\quad \therefore M = \frac{\pi}{32}\cdot f''\cdot\frac{8h^3t + \&c}{h}$

Replacing h by d & neglecting terms involving higher powers of t $\quad M = \frac{\pi}{4}\cdot f''\cdot d^2 t$

But $M = H.Y$

$$\therefore f'' = \frac{4H.Y}{\pi t d^2} = \frac{4H.Y}{Ad}$$

$$\therefore f = \frac{1}{A}\left[P \pm \frac{4H.Y}{d}\right]$$

The sum of the terms gives the max. compression. If the last term is the greater, the difference of the terms gives the max. tension. If the first term is the greater, there is no tension. In order that the stress shall be zero at the side of the pillar where the momt would induce tensn we must have

$$f = \frac{1}{A}\left[P - \frac{4H.Y}{d}\right] = 0 \quad \therefore d = \frac{4H.Y}{P}.$$

C. E. Art. 382.

Here $\frac{c}{x_1}$ = wt. of a unit of length of the chain whose cross-section is sufficient to bear a tension = H. Also length of half span of suspended chain (a parabolic arc)

$$= x_1 + \frac{2}{3}\cdot\frac{y_1^2}{x_1},$$

& as the stress increases with the inclination, if i_1 = inclination at the piers, the stress there

$$= H\sec i_1 = H\sqrt{1+\frac{dy_1^2}{dx_1^2}}$$

Hence

$$c' = \frac{c}{x_1}\left[\left(x_1+\frac{2}{3}\cdot\frac{y_1^2}{x_1}\right)\left(1+\frac{dy_1^2}{dx_1^2}\right)^{1/2}\right] \text{ - - - - - - - - - - - (1)}$$

Substitute for $\frac{dy_1}{dx_1}$ its value from the eq. of the parabola & then expand the term $\left(1+\frac{dy_1^2}{dx_1^2}\right)^{1/2}$ by the binomial theorem

It then $= \left[1+\frac{1}{2}\cdot\frac{4y_1^2}{x_1^2}+\&c\right]$ Substituting in (1) & multiplying

$$c' = c\left(1+\frac{8}{3}\cdot\frac{y_1^2}{x_1^2}+\&c\right)$$

Again, to find c'' –

Let b = half-span. Then $\frac{c}{b}$ = wt. of a unit length of chain fitted to bear a tens.n = H.

Then at any point E in the chain the stress = $H\sec i$ & the wt. of a unit length suited to bear this is

$$\frac{c}{b}\sec i = \frac{c}{b}\left(1+\frac{dy^2}{dx^2}\right)^{1/2}$$

Integrate this for the half span = s & we have

$$c'' = \frac{c}{b}\int\sqrt{1+\frac{dy^2}{dx^2}}\cdot ds = \frac{c}{b}\int\left(1+\frac{dy^2}{dx^2}\right)^{1/2}\left(1+\frac{dy^2}{dx^2}\right)^{1/2}dx$$

$$= \frac{c}{b}\int\left(1+\frac{dy^2}{dx^2}\right)dx = \frac{c}{b}\int\left(1+\frac{x^2}{p^2}\right)dx = \frac{c}{b}\left[x_1+\frac{x_1^3}{3p^2}\right]$$

$$= \frac{c}{b}\left[x_1+\frac{4}{3}\frac{y_1^2}{x_1}\right] = c\left(1+\frac{4}{3}\frac{y_1^2}{x_1^2}\right) \text{ since } b = x_1$$

Eq.s (3) & (4) depend on the tensile strength assigned to the material. Thus in eq. (3)

$$c = \frac{Hx}{4500}$$ Let A = area of cross-section

Then $\frac{C}{12 A x} = \frac{H}{A.5400}$ = wt. of a cub. in. of iron = .277

$\therefore \frac{H}{A} = 54000(.277) = 14958$ lbs which is the tensile strength per sq. in. assumed to obtain eq. (3).

Similarly in eq. (4)

$$\frac{H}{A} = 9972 \text{ lbs.}$$

For eq.s (7), (8), (9), see A. M. Art. 340.

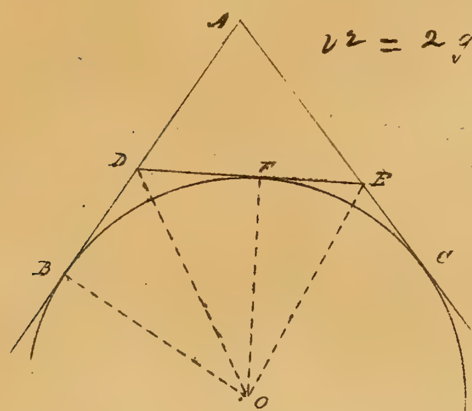
C. E. Art. 431 a

Let g' = retardation due to resistance of brakes, & R = resistance produced by brakes.

Then $\frac{R}{W} = f'$ and

$$W : R :: g : g' \quad \therefore g' = \frac{R}{W} \cdot g = f'g$$

$$v^2 = 2g'h = 2f'gh \quad \therefore h = \frac{v^2}{64.4 f'}$$



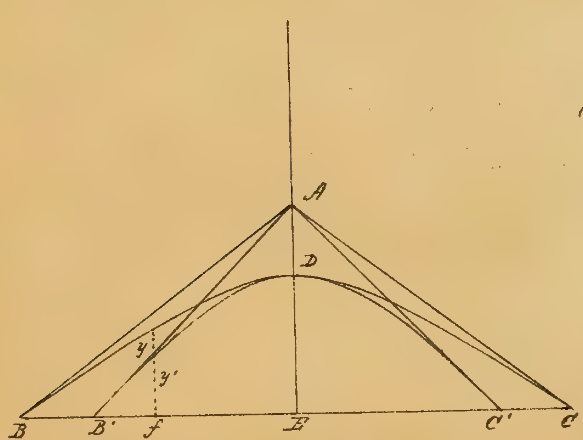
C. E. Art. 434. Prob. I. p. 653

$\angle BOF = \angle ADF \quad \therefore$

$\angle DOF = \angle \tfrac{1}{2} D \quad \therefore DF = FO . \tan \tfrac{1}{2} D$

So $FE = FO . \tan \tfrac{1}{2} E$

$\therefore \quad FO = R = DE \div (\tan \tfrac{1}{2} D + \tan \tfrac{1}{2} E)$



Prob. II. p. 654.

The simple sine curve has for its eq.

$$y = \sin x,$$

& it is represented by $B'DC'$ where $\angle B'AC'$ = a right angle (since $\frac{dy}{dx} = \cos x = 1$ when $x = 0$ which gives $AB'E = B'AE = 45°$). Now, in this curve the radius of curv.

$$= r = \frac{(1+p'^2)^{3/2}}{p''} = \frac{(1+\cos^2 x)^{3/2}}{-\sin x}$$

At D when $x = B'E = 90°$, this $r = 1 = DE$. Again $B'E = AE = 90° = 1.5707$. $DE = 1 \quad \therefore AD = .5707 \quad \therefore \frac{AE}{AD} = 2.75193$

or $AE = 2.75193 . AD$ and $AB' = 2.75193 . AD . \sec . \frac{1}{2} B'AC'$

and $\frac{DE}{AD} = \frac{1}{.5707} = 1.75193$.

Now when the tang.^{s} make any angle as BAC, a "sine curve" may be drawn bet. them by the eq.

$$y = DE . \sin \frac{90° x}{BE}$$

In this case DE no longer = the radius of the circle but is still the max. ordinate; AE remains unchanged, & $BE = 2.75193 . AD . \tan . \frac{1}{2} BAC$. In this case

$$\frac{dy}{dx} = \frac{90° DE}{BE} . \cos \frac{90° x}{BE} \text{ and } \frac{d^2y}{dx^2} = \left(\frac{90°}{BE}\right)^2 . DE . \sin\left(\frac{90° x}{BE}\right).$$

$$\therefore\ r = \frac{\left[1 + \left(\frac{90°}{BE}\right)^2 . \overline{DE}^2 . \cos^2\left(\frac{90° x}{BE}\right)\right]^{3/2}}{\left(\frac{90°}{BE}\right)^2 . DE . \sin\left(\frac{90° x}{BE}\right)} = (\text{when } x = BE \text{ at } D) \frac{1}{\left(\frac{90°}{BE}\right)^2 . DE}$$

$$= \frac{\overline{BE}^2}{DE(1.5707)^2}$$

But $BE = 2.75193 . AD . \tan \frac{1}{2} BAC$ and $AD = \frac{DE}{1.75193}$

$$\therefore\ BE = DE(1.5707) \tan \tfrac{1}{2} BAC$$

$$\text{or } (BE)^2 = \overline{DE}^2 (1.5707)^2 \tan^2 \tfrac{1}{2} BAC.$$

Hence

$$r = DE \tan^2 \tfrac{1}{2} BAC$$

— " —

C. E. Art. 447.

Eq. 1. The coefficient of velocity thro. a thin plate = .97. Substitute this in eq. (3) (Art. 446) calling the theoretical velocity v' & we have

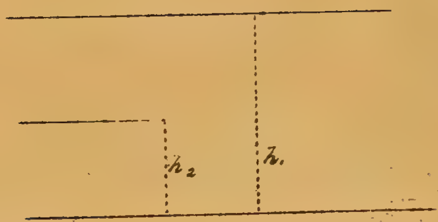
$$.97 v' = 8.025 \sqrt{\frac{v'^2}{64.4(1+F)}}$$

$$\therefore\ F = .054$$

Eq. 2. Here the coefficient of velocity = .813 (since the coefficient of contraction = 1). Substitute in eq. (3) (Art 446) and

$$(.813)^2 = \frac{1}{1+F} \quad \therefore\ F = .505$$

C. E. Art. 449



Here the flow due to that portion of the notch which is not drowned is

$$Q' = 8.025 . c \int_{h_1 - h_2}^{0} b\sqrt{h} . dh$$

$$= 5.35 \, c \, b \, (h_1 - h_2)^{3/2}$$

Of the drowned part, it is

$$Q'' = 8.025 \, c A \sqrt{h} = 8.025 \, c h_2 \, b\sqrt{h_1 - h_2}$$

$$= 5.35 \, c \, b h_2 . \tfrac{3}{2} \sqrt{h_1 - h_2}$$

Add together, & we get

$$Q = 5.35 \, c \, b \left(h_1 + \frac{h_2}{2}\right) \sqrt{h_1 - h_2}$$

C. E. Art. 450

$$v = \frac{Q}{.7854 d^2} = 8.025 \sqrt{\frac{hd}{4fl}} . \qquad \text{From (1)}$$

$$d = \left(\frac{4fl Q^2}{39.73 h}\right)^{1/5}$$

Eq. (7). Let $f'' = f' + f_1$; then $d = d'\left(\frac{f' + f_1}{f'}\right)^{1/5}$

$$= d'\left(1 + \frac{f_1}{f'}\right)^{1/5} = d'\left(1 + \frac{1}{5} . \frac{f_1}{f'}\right) = d'\left(1 + \frac{1}{5} . \frac{f'' - f'}{f'}\right)$$

$$= d'\left(\frac{4}{5} + \frac{1}{5} . \frac{f''}{f'}\right)$$

Eq. (8) $\quad d = d'\left(\frac{h + h''}{h}\right)^{1/5} = d'\left(1 + \frac{h''}{h}\right)^{1/5} = d'\left(1 + \frac{1}{5} . \frac{h''}{h}\right)$

C. E. Art. 451.

We have from eq. (2)

$v' = 8.025 \sqrt{\frac{i . m}{f'}}$ and denoting the required velocity by v'' & the corrected friction by f we have

$$v'' = 8.025 \sqrt{\frac{i . m}{f}}$$

$$\therefore v' : v'' :: \sqrt{\frac{1}{f'}} : \sqrt{\frac{1}{f}} \text{ or } v'' = v' \sqrt{\frac{1}{f}} \div \sqrt{\frac{1}{.007565}}$$

now

$f = f' + f_1$ by making f_1 = difference bet. f & f'.

Also

$$v'' = v'\left[.087 \sqrt{\frac{1}{f}}\right] . \quad \text{But } \sqrt{\frac{1}{f}} = \frac{1}{\sqrt{f' + f_1}} = \frac{1}{\sqrt{.007565 + f_1}}$$

$= \dfrac{1}{.087 + \frac{1}{2}\cdot\frac{f_1}{.087}}$ (since $\sqrt{.007565} + \frac{1}{2}(.007565)^{-\frac{1}{2}} f_1 + \text{\&c}$ equal to the expansion above.

Hence

$$v'' = v'\left[\frac{.087}{.087 + \frac{f_1}{.174}}\right] = v'\left[1 - \frac{f_1}{.174(.087)}\right]$$

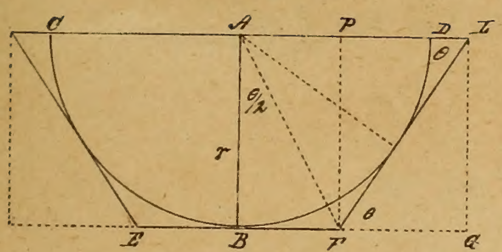
$$= v'\left[1 - \frac{f_1}{.015138}\right].$$

Now for f_1 put its value $f - f' = f - .007565$, & we have

$$v'' = v'\left[1 - \frac{f}{.015138} + \frac{.007565}{.015138}\right] = v'\left[1 - \frac{f}{.015138} + \frac{1}{2}\right]$$

$$= v'\left[\frac{3}{2} - \frac{f}{.015138}\right].$$

Nevilles Section



Angle $BAF = 90^\circ - BFA$

$= 90^\circ - \left(\frac{180^\circ - \theta}{2}\right) = \frac{\theta}{2}$.

$PF \operatorname{cosec}\theta = FI = r \operatorname{cosec}\theta$

Area of $AFI = \frac{1}{2} r^2 \operatorname{cosec}\theta$

" " $ABF = \frac{1}{2} r^2 \tan\frac{\theta}{2}$

Total area $= r^2\left(\operatorname{cosec}\theta + \tan\frac{\theta}{2}\right)$

Wetted border $= 2FI + 2BF$

$= 2r \cdot \operatorname{cosec}\theta + 2r \tan\frac{\theta}{2}$

$$\therefore\ m = \frac{r^2 \operatorname{cosec}\theta + r^2 \tan\frac{\theta}{2}}{2r \operatorname{cosec}\theta + 2r \tan\frac{\theta}{2}} = \frac{1}{2} r$$

$$\therefore\ n = \frac{A}{m^2} = \frac{r^2 \operatorname{cosec}\theta + r^2 \tan\frac{\theta}{2}}{\frac{1}{4} r^2} = 4\left(\operatorname{cosec}\theta + \tan\frac{\theta}{2}\right)$$

Eq. (5) From eq.s (2) & (3) we have

$$m' = \frac{Q'^2}{8512 A^2 i} = (\text{since } A^2 = n^2 m'^4)\ \frac{Q'^2}{n^2 m'^4\, 8512 \cdot i}$$

$$\therefore\ m' = \left(\frac{Q'^2}{8512\, n^2 i}\right)^{1/5} \text{---------} (5)$$

For eq. (7), see eq. (7) (Art. 450)

C. E. Art. 455.

Reservoir with vert. sides. $s = S =$ a constant.

Substituting this value in eq.(3) (Art. 455) we get

$$\frac{t}{T} = \sqrt{h_1}\int_0^{h_1} \frac{b\,dh}{\sqrt{h}} \div \int_0^{h_1} b\,dh = \sqrt{h_1}\int_0^{h_1} b\,h^{-\frac{1}{2}}dh \div \int_0^{h_1} b\,dh$$

$$= \frac{\sqrt{h_1}\,.\,2b\sqrt{h_1}}{bh_1} = 2 \, .$$

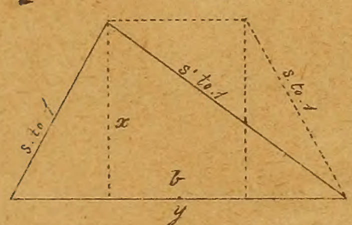
Wedge-shaped reservoir. $s = bh$

$$\frac{t}{T} = \sqrt{h_1}\int_0^{h_1} bh.dh.h^{-\frac{1}{2}} \div \int_0^{h_1} bh\,dh = \frac{\sqrt{h_1}\,.\,\frac{2}{3}h^{3/2}.\,b}{b\frac{h^2}{2}} = \frac{4}{3} = 1\tfrac{1}{3}$$

Pyramidal reservoir. The base of the pyramid at the surface, the apex at the outlet. $s = bh^2$

$$\frac{t}{T} = \sqrt{h_1}\int_0^{h_1} bh^2.h^{-\frac{1}{2}}.dh \div \int_0^{h_1} bh^2 dh = \frac{\sqrt{h_1}\,.\,\frac{2}{5}bh^{5/2}}{b\frac{h^3}{3}} = \frac{6}{5} = 1\tfrac{1}{5}$$

— " —



C. E. Art. 463.

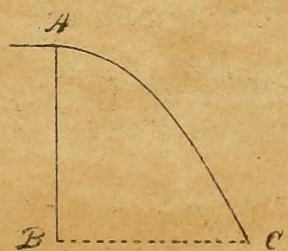
$$b = sx + s'x \quad \therefore \; x = \frac{b}{s+s'}$$

$$y = \frac{s'x - sx}{2} = (s'-s)\frac{x}{2} = \frac{b(s'-s)}{2(s+s')}$$

— " —

C. E. Art. 464.

VI



$$BC = vt = x \; . \quad AB = \tfrac{1}{2}gt^2 = z$$

$$\therefore t = \frac{x}{v} \; ; \quad z = \tfrac{1}{2}g.\frac{x^2}{v^2}$$

$$\therefore x = \frac{2v\sqrt{z}}{\sqrt{2g}}$$

— " —

Note.- For the preparation of these notes for the Lithographer, I am indebted to my young friends and former students - J. Kruttschnitt C.E. and Mr. H. W. Farnham.